Exercises for Unit 3

1. We have seen that any algorithm that finds the minimum and maximum of n keys simultaneously requires at least $\left\lceil \frac{3n}{2} \right\rceil - 2$ comparisons. Devise an algorithm based on the adversarial argument that uses exactly $\left\lceil \frac{3n}{2} \right\rceil - 2$ comparisons.

2. Let $n \geq 3$ be an odd integer. The median of n distinct keys S is defined as the key m with the property that $\left\lfloor \frac{n}{2} \right\rfloor$ of the keys in S are smaller than m, and that $\left\lfloor \frac{n}{2} \right\rfloor$ of the keys in S are larger than m. Use an adversarial argument to show that at least $\frac{3(n-1)}{2}$ comparisons are required to find the median.

   Hint: Partition the keys into four sets (S) the keys smaller than or equal to the median (L) the keys larger than or equal to the median (M) the median (N) the keys for which nothing is known so far.

3. The so-called "$a + b \neq c$"-problem takes as input three sorted sets A, B, C of $\Theta(n)$ numbers each and asks whether for all $a \in A$, $b \in B$, and $c \in C$ we have $a + b \neq c$.

   We will restrict our attention to algorithms for this problem that employ only comparisons (or, ask queries) of the form $x + y < z$, $x + y > z$, or $x + y = z$ with $x \in A$, $y \in B$, and $z \in C$.

   a) Show that this problem can be solved using $O(n^2)$ comparisons.

   b) Use an adversary argument to show that in the worst case we need to ask at least $\Omega(n^2)$ such queries to determine whether indeed no bad triple with $a + b = c$ exists.

      Hint: Consider the three sets $A = \{-10n + 4i | 0 \leq i < n\}$, $B = \{10n + 4i | 0 \leq i < n\}$, and $C = \{4i + 1 | 0 \leq i < 2n\}$. If you pick two numbers a from A and b from B, then there is a unique $c \in C$ that differs only by 1 from $a + b$. What happens if you "perturb" a and b by adding 1/3 to each, and subtracting 1/3 from c? Which queries does this perturbation affect? How can the adversary use this property in order to derive an $\Omega(n^2)$ lower bound?