Algorithms and Data Structures
Winter Term 2015/16
Exercises for Units 21 & 22

1. In contrast to the shortest path, finding the *longest path* between two given nodes s and t in a general graph can be very difficult. If one could find such path easily, it would be possible to efficiently solve the Hamiltonian Path Problem, which is known to be NP-complete.

   Show that in a directed acyclic graph (DAG) the longest directed path between two nodes can be found in time $O(m+n)$.

2. Let G be a directed graph, where each edge is colored either red or blue. Let u and v be two vertices of G.

   a) Design an efficient algorithm to decide whether there exists a directed path from u to v that contains at least twice as many red edges as blue edges.

   b) Design an efficient algorithm for finding a path from u to v that contains as few red edges as possible.

   c) Design an efficient algorithm for finding a path from u to v that contains exactly two red edges, if such a path exists, or otherwise reports failure.

3. Consider the following algorithm for solving the single-source shortest path problem in directed graphs with general edge costs:

```
SSSP(V,E,c,s):
    Improved = {s}
    while Improved ≠ ∅ do
      B = Improved
      Improved = ∅
      while B ≠ ∅
        choose and remove some v from B
        for each w in Out(v) do
          d' = d[v] + c((v,w))
          if d' < d[w] then
            d[w] = d'
            parent[w] = v
            insert w into Improved
```

   Assume `Out(`v`)` denotes the set of out-neighbors of v, and sets `B` and `Improved` are implemented such that insertions, deletions, choosing some element, and test for emptiness only need constant time each.

   a) Does this program always terminate? If no, correct it appropriately.

   b) Show that there are arbitrarily large graphs for which `SSSP()` is asymptotically faster than the Bellman-Ford algorithm

   c) Show that there are arbitrarily large graphs for which `SSSP()` has the same asymptotic running time as the Bellman-Ford algorithm.

4. Due to strikes in German railway companies, Saarstahl is considering to increase the use of trucks for transportation. They want to send trucks from Völklingen to Rotterdam harbor laden as heavily as possible with steel products, but each road segment has a maximum weight limit on trucks that use this road.

   a) Give an efficient algorithm for finding a route that allows to load a truck with maximum possible weight.

   b) An ordered semigroup is a set S together with an operation $\oplus$, a neutral element 0, and a total ordering $\leq$ such that for all $x, y, z \in S$ we have

      - $x \oplus y \in S$,
      - $(x \oplus y) = (y \oplus x)$,
      - $(x \oplus y) \oplus z = x \oplus (y \oplus z)$,
      - $x \oplus 0 = x$, and
      - $x \leq y$ implies $x \oplus z \leq y \oplus z$.

      Show how the Bellman-Ford algorithm can be applied when edge weights in a graph come from an ordered semigroup.

   c) Under which conditions can Dijkstra's algorithm be applied when edge weights in a graph come from an ordered semigroup?