Exercises for Unit 23

1. Explicitly representing the shortest paths between all n^2 vertex pairs of an n-vertex graph may take space up to $\Theta(n^3)$ (since $\Theta(n^2)$ of those paths may consist of $\Theta(n)$ edges each).

   Give a representation that uses space just $O(n^2)$ and that allows to report for any given pair u, v of vertices a shortest path P from u to v in time proportional to the number of edges of P.

2. Let $G = (V, E)$ be a directed graph with vertex set $V = \{1, \ldots, n\}$ and edge costs $c(e)$ for every $e \in E$. Let $D = d(i, j)$, $i, j \in V$, the matrix of shortest path distances, i.e. for $1 \leq i, j \leq n$ the entry $d(i, j)$ equals the length of the shortest path from vertex i to vertex j.

   Now assume a graph G' is obtained from G by adding a vertex $n + 1$ and edge costs $c(e)$ for every edge $e = (1, n + 1), \ldots, (n, n + 1), (n + 1, 1), \ldots, (n + 1, n)$.

   a) How can you obtain the distance matrix D' of G' from the distance matrix D of G? How much time does this take?

   b) Does your method work for negative costs also? If yes, how would you deal with negative cycles?

   c) Can you obtain from your approach a general algorithm for the all pairs shortest path problem? If yes, what is its running time?

   d) What is the relationship of this algorithm to the so-called Floyd-Warshall Algorithm for the all pairs shortest path problem?