Algorithms and Data Structures
Winter Term 2015/16
Exercises for Units 25 & 26

1. Suppose you are given a directed graph $G = (V, E)$, two vertices s and t, a capacity function $c : E \to \mathbb{R}^+$, and another function $f : E \to \mathbb{R}$. Describe an algorithm to determine whether f is a maximum (s,t)-flow in G.

2. Prove the following two versions of Menger's theorem:

   a) Let $G = (V, E)$ be an undirected graph, and let u and v be nonadjacent vertices in G. The maximum number of pairwise edge disjoint u-v-paths in G equals the minimum number of edges from E whose deletion separates u and v.

   b) Let $G = (V, E)$ be an undirected graph, and let u and v be nonadjacent vertices in G. The maximum number of pairwise internal-vertex disjoint u-v-paths in G equals the minimum number of vertices from $V - \{u, v\}$ whose deletion separates u and v.

3. Let (S, T) and (S', T') be minimum (s,t)-cuts in some flow network G. Prove that $(S \cap S', T \cup T')$ and $(S \cup S', T \cap T')$ are also minimum (s,t)-cuts in G.

4. Now consider a flow network $G = (V, E; s, t; c)$ where all capacities are integral and where C is the maximum capacity.

   a) Let f be any legal integral flow for G and consider the residual graph G_f. Let $\overline{f}$ be a maximum flow for G_f.

      Show that G_f admits an augmenting path with flow value at least $|\overline{f}|/m$.

   b) A *maximum capacity augmentation* is one that uses an augmentation path that allows the maximum possible flow value increase (i.e. it is an s–t-path of of largest "bottleneck weight."

      Show that $n + m \ln C$ maximum capacity augmentations suffice to compute a maximum flow for G. Here $n = |V|$ and $m = |E|$.

      Hint: Consider the sequence of maximum flow values in the sequence of residual graphs. What is an upper bound for the first value? How many iterations are needed to reduce it to a value at most n? (Recall: $1 - x \leq e^{-x}$)

   c) What running time do you get for this kind of maximum flow algorithm, assuming that a maximum capacity augmentation can be found in $O(m \log C)$ time?

   d) Show that a maximum capacity augmentation can be determined in $O(m \log C)$ time.