1. Consider a bipartite graph $G = (A \cup B, E)$.
   - Let M_1 and M_2 be two matchings in G. Show that there is always a matching that matches all the vertices of A matched by M_1 and all vertices of B matched by M_2.
   - Let M be a matching in G. Show that there is always a *maximum* matching M^* that matches all the vertices matched by M.

2. Consider the following game. Given an undirected bipartite graph $G = (V, E)$, two players alternately pick distinct vertices $v_1, v_2, \ldots$ from V. For $i = 1, 2, \ldots$ vertex v_{i+1} must be adjacent to vertex v_i. The last player that can pick a vertex wins the game. Show that the first player has a winning strategy if and only if G has no perfect matching.

3. A **vertex cover** A is a subset of the vertices of a graph $G = (V, E)$ such that for all edges at least one endpoint is in A.

   Prove the following Theorem (called König-Egerváry Theorem): In a bipartite graph, the size of a minimum vertex cover equals the size of a maximum matching.

   Hint: you may use the connection of the maximum matching problem to maximum flows and minimum cuts.