1. Consider a single-item auction with $n \geq 3$ bidders and assume that the item is given to the highest bidder. Suppose the payment of the highest bidder is the k-th largest bid, for some $3 \leq k \leq n$. For all other bidders the payment is 0. For which k is this a truthful mechanism? Prove your answer.

2. Consider a 2-connected graph $G = (V, E)$ and a reverse auction as follows. The mechanism should construct a cheap connected network. Each edge $e \in E$ is a bidder with a private cost $c_e \geq 0$ for being in the network, and cost 0 otherwise. Hence, the private valuation of a bidder is $v_e = -c_e \leq 0$ when being in the network and 0 otherwise.

   A mechanism asks bidders for their costs and then purchases a spanning tree of G. Consider the VCG mechanism that buys the usual MST T^* minimizing $\sum_{e \in T^*} c_e$ (= maximizing $\sum_{e \in T^*} v_e$).

   a) Fix distinct values c_e, let T^* denote the MST, and T_0 the cheapest spanning tree edge-disjoint from T^* (assume that such a tree exists). Construct a bipartite graph H with partitions A and B, which represent the edges of T and T_0, respectively.

      An edge (e_1, e_2) is in H if and only if $T^* \setminus \{e_1\} \cup \{e_2\}$ is again a spanning tree of G. Prove that H has a perfect matching.

      Hint: Use Hall's Theorem.

   b) Use the existence of a perfect matching in H to prove that the total payments charged by the VCG mechanism with Clarke pivot payment are at most the cost of the cheapest spanning tree T_0 that is edge-disjoint from MST T^*.

3. Consider a combinatorial auction where you know a priori that every bidder is *unit demand*: The valuation of a bidder i can be described by m non-negative private parameters (one per item) $v_{i1}, \ldots, v_{im} \geq 0$. For an arbitrary subset S of items we compose these values by $v_i(S) = \max_{j \in S} v_{ij}$. Prove that the VCG mechanism can be implemented in polynomial time for unit-demand bidders.

4. Consider adjusted versions of the greedy mechanism for single-minded combinatorial auctions. To construct an allocation they use different orderings to greedily add bidders to the winner set (assume there is a consistent tie-breaking):

   - $v_1^t \geq \ldots \geq v_n^t$
   - $|S_1^t| \geq \ldots \geq |S_n^t|$
   - $v_1^t/|S_1^t| \geq \ldots \geq v_n^t/|S_n^t|$

   a) Which of these allocation algorithms can be turned into truthful mechanisms using suitable payments and why?

   b) Prove that it is possible to bound the approximation ratio in terms of the number of items m or provide a counterexample.

   c) Provide a tight example for the approximation ratio in each case when bounding in terms of m is possible.