Mechanism Design

Algorithms and Data Structures

Winter 2016

Mechanism Design (with Money)

- Set A of outcomes to choose from.
- n bidders with quantifiable preferences.
- Preference of bidder i is given by a valuation function $v_i : A \to \mathbb{R}$ from a commonly known set $V_i \subseteq \mathbb{R}^A$
- A mechanism determines an outcome $a \in A$ and payments, charges each bidder some amount m_i of money
- Utility of bidder i is $v_i(a) - m_i$, *quasi-linear* utilities.
- Common currency enables utility transfer between bidders.

Example: Sealed Bid Auction

A single item is sold to one customer.

Customer	1	2	3	4	5
Value	9	1	20	11	14

Agents report their values as sealed bid.
Social Choice: Winner is agent with highest bid.
Payments: find payments to ensure truthful bidding

- No payments: Bidders strive to bid unboundedly high values.
- Payments = Bids: Bidders try to guess the second highest bid and bid a slightly higher value.

Vickrey Second Price Auction

Payment of the winner is the second largest bid.

Value	9	1	20	11	14
Payment	0	0	14	0	0
Utility	0	0	6	0	0

Proposition

The Vickrey auction is truthful.

Example

Value	?	?	20	?	?
Bid	5	11	x	2	14
Payment			14		
Utility			6		

Case 1: i wins with true value $x = 20$, then for all $x \geq 14$ utility 6, for $x < 14$ utility 0.

Value	?	?	20	?	?
Bid	5	11	x	2	24
Payment			0		
Utility			0		

Case 2: i loses with true value $x = 20$, then for all $x < 24$ utility 0, for $x \geq 24$ utility -4.

Vickrey Auction

Vickrey-Clarke-Groves Mechanisms

Single-Minded Combinatorial Auctions

Definitions

Direct Revelation Mechanism

- Denote $V = V_1 \times \ldots \times V_n$ and $v \in V$
- Social choice function $f : V \to A$
- A vector of payment functions $p_1, \ldots, p_n$
- Each function $p_i : V \to \mathbb{R}$ specifies the amount bidder i pays.

Truthfulness / Incentive Compatibility (IC)

- For every bidder i, profile $v \in V$, alternative $v_i' \in V_i$,
- Denote outcomes by $a = f(v_i, v_{-i})$ and $b = f(v_i', v_{-i})$
- Mechanism $(f, p_1, \ldots, p_n)$ is truthful (or incentive compatible) if the utility

$$v_i(a) - p_i(v_i, v_{-i}) \geq v_i(b) - p_i(v_i', v_{-i})$$

.

Sealed-Bid Auction

Bidder	1	2	3	4	5
Value	9	1	20	11	14

- Outcomes $A = \{1, 2, 3, 4, 5\}$, where i means "i wins"

Outcome	1	2	3	4	5
v_1	9	0	0	0	0
v_2	0	1	0	0	0
etc.					

- Social Choice: $f(v) = \arg\max_i \{v_i(i)\}$
- Payments: $p_i(v) = 0$ if $f(v) \neq i$,
  otherwise $p_i(v) = \max_{j \neq i} v_j(j)$.

VCG Mechanism

Definition

A Vickrey-Clarke-Groves (VCG) mechanism is given by

- $f(v) \in \arg\max_{a \in A} \sum_i v_i(a)$, and
- for some functions $h_1, \dots, h_n$ with $h_i : V_{-i} \to \mathbb{R}$ and all $v \in V$ we have

$$p_i(v) = h_i(v_{-i}) - \sum_{j \neq i} v_j(f(v))$$

Observations:

- VCG mechanism picks outcome a that maximizes social welfare $\sum_j v_j(a)$
- h_i does not depend on the "own bid" v_i
- Utility of bidder i when $f(v) = a$:

$$v_i(a) - p_i(v) = \sum_j v_j(a) - h_i(v_{-i})$$

VCG is IC

Theorem

Every VCG mechanism is truthful.

Proof:

- Given types v, for bidder i a "lie" v_i', and let $a = f(v)$ and $b = f(v_i', v_{-i})$.
- Utility of i declaring v_i is $v_i(a) + \sum_{j \neq i} v_j(a) - h_i(v_{-i})$
- Utility of i declaring v_i' is $v_i(b) + \sum_{j \neq i} v_j(b) - h_i(v_{-i})$
- Utility is maximized when outcome maximizes social welfare $\sum_j v_j(x)$.
- VCG mechanism maximizes social welfare, $\sum_j v_j(a) \geq \sum_j v_j(b)$.
- By declaring v_i' bidder i, VCG picks b which is optimized for his lie, but possibly suboptimal for the real utility.
- VCG aligns each bidder incentive with the social incentives. □

Desirable Properties of Payments

Definition

- A mechanism is (ex-post) individually rational if bidders always get non-negative utility, i.e. for all $v \in V$ we have $v_i(f(v)) - p_i(v) \geq 0$.
- A mechanism has no positive transfers if no bidder is ever paid money, i.e. for all $v \in V$ and all i we have $p_i(v) \geq 0$.

Definition (Clarke Rule)

The choice $h_i(v_{-i}) = \max_{b \in A} \sum_{j \neq i} v_j(b)$ is called Clarke pivot payment.

Then the payment of bidder i becomes

$$p_i(v) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(f(v))$$

Payment is the total damage to the other bidders caused by the presence of i. Each bidder *internalizes externalities*.

Clarke Rule

Lemma

A VCG mechanism with Clarke pivot payments makes no positive transfers. If $v_i(a) \geq 0$ for all $v_i \in V_i$ and $a \in A$, then it is individually rational.

Proof:

- Let $a = f(v)$ and $b = \arg\max_{a' \in A} \sum_{j \neq i} v_j(a')$
- No positive transfers (by definition)

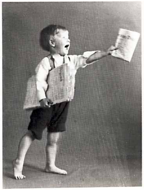
$$\sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a) \geq 0$$

- Individually rational

$$v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b) \geq \sum_j v_j(a) - \sum_j v_j(b) \geq 0$$

□

Example: Bilateral Trade



	trade	no-trade
Seller	$-v_s$	0
Buyer	v_b	0

- Trade occurs if $v_b > v_s$, no-trade if $v_s > v_b$
- Analyze VCG Mechanism, should not subsidize trade.

Example: Bilateral Trade

	trade	no-trade
Seller	$-v_s$	0
Buyer	v_b	0

- VCG payments for no-trade:
  Seller payments: $h_s(v_b) - 0$, Buyer payments: $h_b(v_s) - 0$
  No additional payments by the mechansims, so $h_s(v_b) = h_b(v_s) = 0$.
- VCG payments for trade:
  Seller payments: $h_s(v_b) - v_b$, Buyer payments: $h_b(v_s) + v_s$
  Seller receives v_b, but buyer pays only $v_s < v_b$.
- Not *budget-balanced*: VCG mechanism subsidizes trade!

Example: Procurement or Reverse Auction

- Auctioneer buys service, participants offer service for a cost
- Auctioneer pays participants
- Negative utility, negative payments
- Vickrey auction:
  Pick cheapest participant, pay second smallest offered cost

Corollary

The Vickrey reverse auction is incentive compatible.

Vickrey Reverse Auction is IC

Case 1: If bidding his true value, bidder i wins.

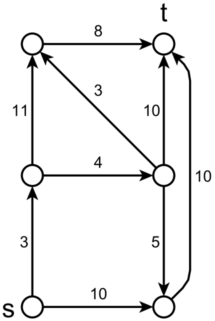
Value	?	?	-7	?	?
Bid	-9	-11	x	-17	-14
Payment			-9		
Utility			2		

Case 2: If bidding his true value, bidder i loses.

Value	?	?	-12	?	?
Bid	-9	-11	x	-17	-24
Payment			0		
Utility			0		

Example: Buying a Path in a Network

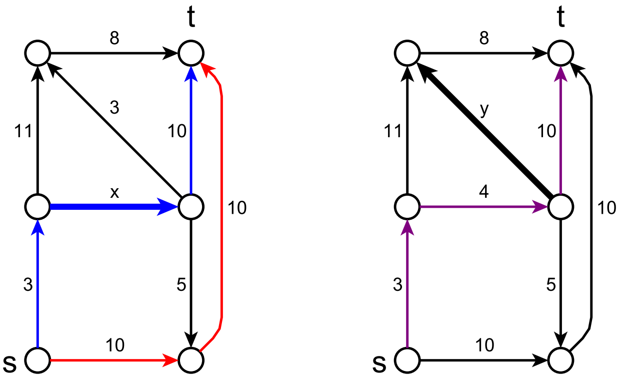
Reverse auction:
Bidders are edges in a network. Mechanism needs to buy an s-t-Path.



Example: Buying a Path in a Network

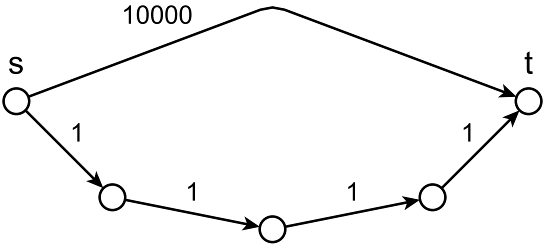
- Outcomes are s-t-paths in graph G
- VCG picks the shortest path P^* for reported costs c_e
- Payments for $e \in P^*$ are $h_e(c_{-e}) + \sum_{e' \neq e} c_{e'}(P^*)$
- Clarke pivot payment: $h_e(c_{-e}) = -\min_{P \in G-e} \sum_{e \in P} c_e$
- Total payment $c(P^* - e) - c(P^*_{-e})$, where P^*_{-e} is shortest path in G when it would not contain e.
- An edge $e \notin P^*$ gets no payment.

Truth-telling is a dominant strategy!



Frugality

Incentive compatibility might be EXPENSIVE!

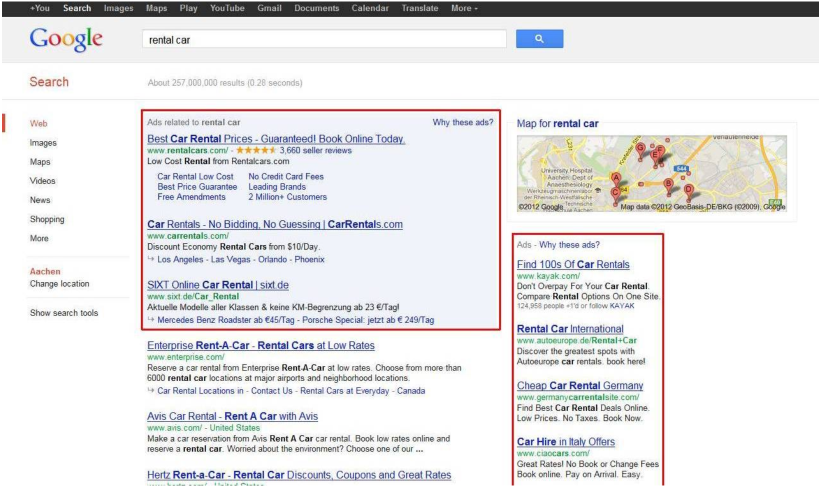


Vickrey Auction

Vickrey-Clarke-Groves Mechanisms

Single-Minded Combinatorial Auctions

Ad Auctions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General Setting

Combinatorial Auction:

- Set M of m indivisible items (e.g., ad slots) auctioned simultaneously
- n bidders, valuations for each subset of items
- Who should get which items and pay how much?
- *General Allocation Problem of Interrelated Resources*

Valuation v_i for bidder i:

- $v_i(S) \in \mathbb{R}$ when getting assigned set $S \subseteq M$
- free disposal: $S \subseteq T \Rightarrow v(S) \leq v(T)$
- normalized: $v(\emptyset) = 0$.

Allocation

- Allocation of the items:
  $S_1, \ldots, S_n$ where $\bigcup_i S_i \subseteq M$ and $S_i \cap S_j = \emptyset$ for $i \neq j$.
- Valuation of a bidder independent of items received by other bidders (no externalities)
- Social Welfare: $\sum_i v_i(S_i)$.
  An optimal allocation $S_1^*, \ldots, S_n^*$ maximizes social welfare.
- Quasi-linear utilities: $v_i(S_i) - p_i(v_i, v_{-i})$
- VCG is truthful with S^*, but computing S^* is NP-hard!
- Let us restrict attention to a special case.

Single-Minded Valuations

Definition

A valuation v_i is single-minded if there exists a threshold bundle S^t and value $v^t \in \mathbb{R}^+$ such that $v_i(S) = v^t$ for all $S \supseteq S^t$, and $v_i(S) = 0$ otherwise.

A single-minded bid is (S^t, v^t), and a bidder can lie about both v^t and S^t.

Definition

The allocation problem among single-minded bidders is given by:

INPUT: (S_i^t, v_i^t) for each bidder $i = 1, \ldots, n$

OUTPUT: Set of winners $W \subseteq \{1, \ldots, n\}$ with maximum social welfare $\sum_{i \in W} v_i^t$ and such that $S_i^t \cap S_j^t = \emptyset$ for each $i, j \in W$ with $i \neq j$

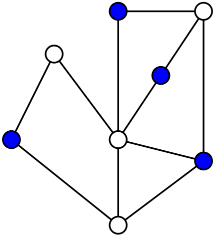
Reduction from Independent Set

Theorem

The allocation problem among single-minded bidders is NP-hard.

Proof:

INDEPENDENT SET: Has a graph an independent set of size at least k?



Vertices $\rightarrow$ Bidders, Edges $\rightarrow$ Items

$(S_i^t, v_i^t) = (\text{Set of incident edges}, 1)$ □

Approximation Algorithms

- c-approximation algorithm: Returns an allocation T such that with the optimal allocation S^* we have

$$\sum_i v_i(T_i) \geq \frac{\sum_i v_i(S_i^*)}{c}$$

- Simple n-approximation algorithm:
  Bidder with the maximum valuation gets M.
  Trivially yields an IC mechanism, essentially single-item VCG auction.

Theorem

For any $\epsilon > 0$ it is NP-hard to approximate INDEPENDENT SET to within a factor of $n^{1-\epsilon}$.

Corollary

For any $\epsilon > 0$ it is NP-hard to approximate the allocation problem among single-minded bidders to within a factor of $n^{1-\epsilon}$.

Approximation Algorithms

The graph in the reduction has at most $m < n^2$ edges/items, hence

Proposition

For any $\epsilon > 0$ it is NP-hard to approximate the allocation problem to within a factor of $m^{1/2-\epsilon}$.

Note: $\sqrt{m} < n$ for sparse instances.

So far, our best algorithm yields an n-approximation. Can we get a truthful mechanism that returns an allocation that is a $\sqrt{m}$-approximation?

Greedy Mechanism for Single-Minded Bidders

INPUT: (S_i^t, v_i^t) for each bidder i

OUTPUT: A set of winners W, payments p_j for all $1 \leq j \leq n$.

Initialization:

1. Reorder bids: $\frac{v_1^t}{\sqrt{|S_1^t|}} \geq \ldots \geq \frac{v_n^t}{\sqrt{|S_n^t|}}$
2. $W \leftarrow \emptyset$, $p_i = 0$ for all i

Iteration:

3. For $i = 1 \ldots n$ do: If $S_i^t \cap \left(\bigcup_{j \in W} S_j^t\right) = \emptyset$ then $W \leftarrow W \cup \{i\}$

Payments:

4. For each $i \in W$ do
5. find smallest index j such that $S_i^t \cap S_j^t \neq \emptyset$ and for all $k < j, k \neq i$ it holds $S_k^t \cap S_j^t = \emptyset$
6. if j exists, set $p_i = \frac{v_j^t}{\sqrt{|S_j^t|/|S_i^t|}}$

Example

Phone	Headset	Power		Mary	John	Jack
x				50	0	0
	x			0	0	0
		x		0	0	0
x	x			50	60	0
x		x		50	0	65
	x	x		0	0	0
x	x	x		50	60	65

Example

Reordering:

	S_i^t	v_i^t	$v_i^t/\sqrt{\lvert S_i^t \rvert}$
1. Mary	Phone	50	50
2. Jack	Phone, Power	65	45.96...
3. John	Phone, Headset	60	42.42...

Algorithm determines W and p_i :

- 1. Mary: $W = \emptyset$, so $W = \{1\}$
- 2. Jack: $S_1^t \cap S_2^t = \{\text{Phone}\}$
- 3. John: $S_1^t \cap S_3^t = \{\text{Phone}\}$
- Winner is Mary

- First bidder blocked by Mary, which could be in W, is Jack (2)
- Payments: $p_1 = v_2^t/\sqrt{|S_2^t|/|S_1^t|} = 65/\sqrt{2/1} = 45.96...$

Incentive Compatibility

Lemma

A mechanism for single-minded bidders with $p_i = 0$ whenever $i \notin W$ is IC if and only if for every bidder i and fixed other bids (S^t_{-i}, v^t_{-i}) the following holds:

- *Monotonicity: If bidder i wins with (S^t_i, v^t_i), then he remains a winner for any $v'_i > v^t_i$ and $S'_i \subset S^t_i$.*
- *Critical Payment: A winning bidder pays the minimum value needed for winning – the infimum of all values v'_i such that (S^t_i, v'_i) still wins.*

Greedy is IC

Monotonicity: (S_i^t, v_i^t) wins, then it wins with any $v_i' > v_i^t$ and $S_i' \subset S_i^t$
Critical Payment: Winner pays infimum of all v_i' such that (S_i^t, v_i') wins.

Does Greedy satisfy it?

- Increasing v_i^t or reducing S_i^t increases $v_i^t / \sqrt{|S_i^t|}$
- i moves up in order and remains winning
- Payment is the switching point between i and j:

$$\frac{x}{\sqrt{|S_i^t|}} \leq \frac{v_j^t}{\sqrt{|S_j^t|}} \quad \Rightarrow \quad x \leq v_j^t \frac{\sqrt{|S_i^t|}}{\sqrt{|S_j^t|}} = \frac{v_j^t}{\sqrt{|S_j^t|/|S_i^t|}}$$

Proof of Lemma (if-part)

Initial Observations

- Truthful bidder has always positive utility
- Bidder has (S, v) and bids $(S', v') \neq (S, v)$
- If (S', v') is a losing bid, reporting (S, v) can only help.
- If $S \not\subseteq S'$, reporting (S, v') can only help.

Assumption: (S', v') is winning bid and $S \subseteq S'$.

Winner is never worse off to bid (S, v'):

- Denote payment p' for (S', v') and p for (S, v').
- If (S, x) with $x < p$ loses, then (monotone) (S', x) loses.
- Thus, for the critical payments $p' \geq p$.
- (S, v') causes at most the payments of (S', v').
  It can win in cases, in which (S', v') loses.

Proof of Lemma (if-part)

If bidders reveal their true sets S, truthful bidding of v follows:

- Assume (S, v') wins and (S, v) also
- Critical payment p for (S, v)
- For $v' > p$ same payments, for $v' < p$ losing $\Rightarrow$ IC

- Assume (S, v') wins and (S, v) loses
- v smaller than critical payments, negative utility for (S, v') $\square$

Approximation of Social Welfare

Lemma

The greedy mechanism computes a $\sqrt{m}$-approximation for the corresponding allocation problem.

Proof:

- Denote optimal winner set W^*, output of greedy W.
- For each $i \in W$ consider $W_i^* = \{j \in W^*, j \geq i \mid S_j^t \cap S_i^t \neq \emptyset\}$.
- Every $j \in W^*$ appears in at least one W_i^*, so $\sum_i \sum_{j \in W_i^*} v_j^t \geq \sum_{i \in W^*} v_i^t$.
- Claim: $\displaystyle\sum_{j \in W_i^*} v_j^t \leq v_i^t \sqrt{m}$.
- Then lemma follows with intuitive accounting argument:
  Consider value that greedy loses compared to optimum because of adding i to W. This is at most a factor of $\sqrt{m}$ larger than the value he secures by adding i to W.

Proving the Claim

For every $j \in W_i^*$ we have $j \geq i$, and so by the order

$$\frac{v_j^t}{\sqrt{|S_j^t|}} \leq \frac{v_i^t}{\sqrt{|S_i^t|}} \quad \Rightarrow \quad v_j^t \leq \frac{v_i^t \sqrt{|S_j^t|}}{\sqrt{|S_i^t|}} \ .$$

Summing over all $j \in W_i^*$ we get

$$\sum_{j \in W_i^*} v_j^t \leq \frac{v_i^t}{\sqrt{|S_i^t|}} \sum_{j \in W_i^*} \sqrt{|S_j^t|} \ .$$

The following Cauchy-Schwartz inequality

$$\left(\sum_{j \in W_i^*} 1 \cdot \sqrt{|S_j^t|} \right)^2 \leq \left(\sum_{j \in W_i^*} 1^2 \right) \cdot \left(\sum_{j \in W_i^*} \left(\sqrt{|S_j^t|} \right)^2 \right)$$

yields a bound on the last term:

$$\sum_{j \in W_i^*} \sqrt{|S_j^t|} \leq \sqrt{|W_i^*|} \sqrt{\sum_{j \in W_i^*} |S_j^t|} \ .$$

Proving the Claim

Combining the last two bounds we have so far:

$$\sum_{j \in W_i^*} v_j^t \leq \frac{v_i^t}{\sqrt{|S_i^t|}} \sqrt{|W_i^*|} \sqrt{\sum_{j \in W_i^*} |S_j^t|} .$$

- Every S_j^t intersects S_i^t for $j \in W_i^*$.
- W^* yields allocation, so $S_j^t \cap S_k^t = \emptyset$ for $j, k \in W_i^*$
- This means $|W_i^*| \leq |S_i^t|$.
- W^* is allocation, so $\sum_{j \in W^*} |S_j^t| \leq m$

This gives

$$\sum_{j \in W_i^*} v_j^t \leq v_i^t \sqrt{\sum_{j \in W^*} |S_j^t|} \leq v_i^t \sqrt{m}$$

and finishes the proof. □