Exercises for Unit 32

Problem 1.

Define $P_A(x) = \sum_{i=1}^{k} m_i x^i$, where m_i = # times i appears in A. , $\deg P_A = k$

and similarly $P_B(x) = \sum_{i=1}^{k} m_i' x^i$ —"— m_i' = ——"—— B.

Let $P(x) = P_A(x) - P_B(x)$.

Then $A = B \Leftrightarrow P(x) \equiv 0$. ⤳ 0 polynomial

Method

- Pick set $U \subset \mathbb{R}$
- Pick c variables $x_1, x_2, \dots, x_c$ with $x_i \in U$, $i \in \{1, \dots, c\}$.
- Compute $P(x_1), \dots, P(x_c)$.
- If $\exists i : P(x_i) \neq 0$, $i \in \{1, \dots, c\}$ then return A, B distinct

  Else return A, B are the same.

Running time: evaluating polynomial $P(x_i)$ takes time $O(k)$ ⇒ total running time $O(k \cdot c)$.

~~If A, B are distinct we return the correct ans~~

What is the probability that we return a wrong answer?

This can happen when A, B are distinct but we return they are the same:

So $P \not\equiv 0$, but $P(x_1) = P(x_2) \dots = P(x_c) = 0$.

However: $\Pr(P(x_1) = 0) \leq \frac{k}{|U|}$.

$\Rightarrow \Pr(P(x_1) = 0 \wedge P(x_2) = 0 \wedge \dots P(x_c) = 0) \leq \left(\frac{k}{|U|}\right)^c$.

Picking $|U|, c$ large decreases the probability we are wrong.

Note: different solution:

We could pick $P_A(\vec{x}) = \sum_{i=1}^{k} m_i x_i$, $P_B(\vec{x}) = \sum_{i=1}^{k} m_i' x_i$. $\deg P_A \leq 1$

Then the variables $v_1, v_2, \dots, v_c$ would be: $v_i \in U^k$, $i \in \{1, \dots, c\}$.

The probability we are wrong is:

$$\Pr(P(v_i) = 0) \leq \frac{1}{|U|} \Rightarrow \Pr(P(v_1) = 0 \text{ and} \dots P(v_c) = 0) \leq \frac{1}{|U|^c}$$

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(2.) induction over n

$\boxed{n=1}$ $P(x_1) = a_k x_1^k + a_{k-1} \cdot x_1^{k-1} + \dots + a_1 x_1 + a_0 \cdot x_1^0$

positive roots $\leq k+1-1 = k$

$\Rightarrow \Pr(P=0) \leq \frac{k}{L}$ ✓

$\boxed{n>1}$ $P(x_n, x_{n-1}, \dots, x_1) = \sum_{i=0}^{m} x_n^i \cdot g_i(x_{n-1}, \dots, x_1) = (*)$

m = highest power with which x_n occurs

$$(*) = \underbrace{x_n^m \cdot g_m(x_{n-1}, \dots, x_1)}_{l_1 \text{ monomials}} + \underbrace{\sum_{i=0}^{m-1} x_n^i \cdot g_i(x_{n-1}, \dots, x_1)}_{l_2 \text{ monomials}} \qquad l_1 + l_2 = k+1$$

$$\Pr(P(\vec{x})=0) = \Pr(P(\vec{x})=0 \wedge g_m(x_{n-1}, \dots, x_1)=0)$$
$$+ \Pr(P(\vec{x})=0 \wedge g_m(x_{n-1}, \dots, x_1) \neq 0)$$

$$= \Pr(P(\vec{x})=0 \wedge g_m(x_{n-1}, \dots, x_1)=0) + \Pr(P(\vec{x})=0 \mid g_m(x_{n-1}, \dots, x_1) \neq 0) \cdot \Pr(g_m(x_{n-1}, \dots, x_1) \neq 0)$$

$$\leq \Pr(g_m(x_{n-1}, \dots, x_1)=0) + \underbrace{\Pr(P(\vec{x})=0 \mid g_m(x_{n-1}, \dots, x_1) \neq 0)}_{\text{apply base case}}$$

$$\leq \frac{l_1 - 1}{L} + \frac{(l_2+1)-1}{L} = \frac{k}{L}$$

apply base case
↳ know that the leading coeff. in $(*)$ is not 0, so when we look at it as poly in one variable, x_n, this poly has $\leq l_2 + 1$ monomials
$\Rightarrow \Pr(f(x_n)=0) \leq \frac{(l_2+1)-1}{L}$

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Problem 3

We note that a matching is a perfect matching when restricted to the vertices that are matched.

Let M be a maximum matching of G, and $G' = (V', E')$ the restriction of G to the vertices in M.
Then Edmonds matrix of G': $A_{G'}$ has nonzero determinant

Hower $A_{G'}$ is a submatrix of A of size $|M|$ $\Rightarrow$ rank $A \geq |M|$.

Assume rank $A > |M|$ $\Rightarrow \exists$ submatrix of A called A' with $\det A' \neq 0$ and size $>|M|$. However, A' is the Edmonds matrix of a subgraph G' of G with size $>|M|$ $\Rightarrow$ G' has a perfect matching $\Rightarrow G$ has a matching of size $>|M|$ $\lightning$.

$\Rightarrow$ rank $A = |M|$.