Exercises for Unit 36

Problem 1

$f \to$ equilibrium flow

$g \to$ any other flow.

We show first that $\sum_{P \in \mathcal{P}} f_P \ell_P(f) \leq \sum_{P \in \mathcal{P}} g_P \ell_P(f)$. $(*)$

f equilibrium $\Rightarrow$ $\ell_P(f) = \ell$ if $P \in \mathcal{P}'$, $\mathcal{P}'$ has paths where $f_P > 0$.

$\ell_P(f) \geq \ell$ if $P \notin \mathcal{P}'$, so $f_P = 0$.

$$\left.\begin{aligned} \sum_{P \in \mathcal{P}} g_P \ell_P(f) &= \sum_{P \in \mathcal{P}'} g_P \ell_P(f) + \sum_{P \notin \mathcal{P}'} g_P \ell_P(f) \geq \ell \cdot \sum_{P \in \mathcal{P}} g_P = \ell. \\ \sum_{P \in \mathcal{P}} f_P \ell_P(f) &= \sum_{P \in \mathcal{P}'} f_P \underbrace{\ell_P(f)}_{\ell} + \sum_{P \notin \mathcal{P}'} \underbrace{f_P}_{0} \ell_P(f) \leq \sum_{P \in \mathcal{P}'} f_P \cdot \ell \leq \ell \end{aligned}\right\} \Rightarrow (*).$$

Now: $C(f) = \sum_{P \in \mathcal{P}} f_P \ell_P(f) \leq \sum_{P \in \mathcal{P}} g_P \ell_P(f) = \sum_{e \in E} g_e \ell_e(f_e) = \sum_{e \in E} g_e(\ell_e(g_e) + \ell_e(f_e) - \ell_e(g_e))$

$= C(g) + \sum_{e \in E} g_e(\ell_e(f_e) - \ell_e(g_e))$.

<u>Lemma</u> For every $e \in E$, $g_e(\ell_e(f_e) - \ell_e(g_e)) \leq \phi f_e \ell_e(f_e)$, where $\phi = \frac{2}{3\sqrt{3}}$.

Then: $C(f) \leq C(g) + \sum_{e \in E} \phi f_e \ell_e(f_e) = C(g) + \phi C(f) \Rightarrow C(f)(1 - \phi) \leq C(g)$

$\Rightarrow \frac{C(f)}{C(g)} \leq \frac{1}{1-\phi} \Rightarrow$ P.O.A is $\frac{1}{1-\phi}$.

Proof of Lemma. $\ell_e(f_e) = ax^2 + bx + c$, $a, b, c > 0$ (since ℓ_e is nonnegative and increasing).

We show statement for $\ell_e(f_e) = x^2$ wlog.

Need to show: $g_e(f_e^2 - g_e^2) \leq \phi f_e \cdot f_e^2$ / divide by f_e^3:

$$\frac{g_e}{f_e} - \frac{g_e^3}{f_e^3} \leq \phi, \quad \text{let } x = \frac{g_e}{f_e} \Rightarrow x \in (0,1).$$

Need $x - x^3 \leq \phi$.

$(x - x^3)$ is maximum when derivative is $0 \Rightarrow (x - x^3)' = 0 \Rightarrow 1 - 3x^2 = 0 \Rightarrow x = \frac{1}{\sqrt{3}}$.

Max value of $x - x^3$ is then $\frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{2}{3\sqrt{3}} \Rightarrow x - x^3 \leq \frac{2}{3\sqrt{3}}$ ∎

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2) <u>Note</u> : Solution here outlines the main intuition behind why the result should be true. Making technically accurate arguments is beyond the scope of the course and is not the intended goal of this exercise.

For simplicity, consider the case when there is a single source and dest^n.

Consider any feasible flow f in the system. According to function $\phi(\cdot)$

$$\phi(f) = \sum_{e \in E} \frac{a_e}{2} \cdot f_e^2 + b_e \cdot f_e$$

We will think of this function as a multi variate function. Pick two paths P_1 and P_2 on which there is flow (we assume there exists two such paths).

WLOG, assume that the partial derivative of ϕ wrt the edges on path P_1 is greater than that on path P_2.

- Intuitively, this means that increasing the flow by Δf on path P_1 increases $\phi(\cdot)$ more than if we increase the flow by Δf on path P_2.

- This implies that we could transfer infinitesimal flow from P_1 to P_2 and this will decrease $\phi(\cdot)$

- With this argument, we can now claim that if $\phi(\cdot)$ is min for some flow f^*, then it must be the case that the partial der of $\phi(\cdot)$ on all paths is the same.

- However, partial derivative of ϕ along any path corresponds to the latency of this path i.e $\ell_p(f^*)$

- Since the latency on all paths is same, it must be the case that f^* is in Wardrop Equilibrium.

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Problem 2

Solution using convex optimization ~> beyond the scope of this class.

Let $f = (f_{P_1}, f_{P_2}, \dots, f_{P_k}, f_{e_1}, f_{e_2}, \dots, f_{e_m})$ be a flow; $\forall P_i \in \mathcal{P}, \forall e_j \in E$.

We look at the case with only one source-destination pair.

The conditions that f is feasible can be written as:

$$-\sum_{P\in\mathcal{P}} f_P + 1 = 0 \quad (i)$$

$$-f_e + \sum_{e\in P} f_P = 0 \quad \forall e \in E \ (ii)$$

$$-f_P \le 0 \quad \forall P \in \mathcal{P} \ (iii)$$

and we need to minimize $\phi(f) = \sum_{e\in E} \frac{a_e}{2} f_e^2 + b_e f_e$.

-> this is a convex program => f is optimum iff. it satisfies the Karush-Kuhn-Tucker conditions:

(=) ∃ constants λ_P $\forall P \in \mathcal{P}$ corresponding to condition (iii)

λ ~> condition (i)

λ_e $\forall e \in E$ ~> conditions (ii)

such that the following conditions hold:

- primal feasibility (=) f feasible flow
- stationarity: $\nabla \phi(f) + \lambda \nabla\left(-\sum_{P\in\mathcal{P}} f_P + 1\right) + \sum_{e\in E} \lambda_e \nabla\left(-f_e + \sum_{e\in P} f_P\right) + \sum_{P\in\mathcal{P}} \lambda_P \nabla(-f_P)$.

~> this means the variables $\lambda, \lambda_P, \lambda_e$ encode how much ϕ would increase if their respective constraint were relaxed.

$$\nabla f = \left(\frac{\partial f}{\partial f_{P_1}}, \dots, \frac{\partial f}{\partial f_{P_k}}, \frac{\partial f}{\partial f_{e_1}}, \dots, \frac{\partial f}{\partial f_{e_m}}\right).$$

Look at entry corresponding to ∂f_e =>

$$\frac{\partial \phi(f)}{\partial f_e} + \lambda \frac{\partial(-\sum f_P + 1)}{\partial f_e} + \sum_{e'\in E} \lambda_{e'} \frac{\partial(-f_{e'} + \sum_{e'\in P} f_P)}{\partial e} + \sum_{P\in\mathcal{P}} \lambda_P \frac{\partial(-f_P)}{\partial f_e} = 0 \Rightarrow$$

$$\Rightarrow a_e f_e + b_e + \lambda \cdot 0 + \lambda_e(-1) + 0 \Rightarrow \lambda_e = a_e f_e + b_e \Rightarrow \lambda_e \text{ is the latency of the edge } e.$$

Entry corresponding to ∂f_P =>

$$\frac{\partial \phi(f)}{\partial f_P} + \lambda \frac{\partial(-\sum_{P'\in\mathcal{P}} f_{P'} + 1)}{\partial f_P} + \sum_{e\in E} \lambda_e \frac{\partial(-f_e + \sum_{e\in P'} f_{P'})}{\partial P} + \sum_{P'\in\mathcal{P}} \lambda_{P'} \frac{\partial(-f_{P'})}{\partial f_P} \Rightarrow$$

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Problem 2 Part 2

$$\Rightarrow \quad 0 \bullet \lambda \cdot 1 + \sum_{e\in p} \lambda_e + \lambda_p^{\cdot(-1)} = 0 \quad \Rightarrow \sum_{e\in p} \lambda_e = \lambda_p + \lambda$$

- dual feasibility: $\lambda_p \geq 0$ $\leadsto$ it means (iii) holds.

  Then: $\sum_{e\in p} \lambda_e \geq \lambda$, so λ represents the minimal latency of a path.

- complementary slackness: $\lambda_p f_p = 0 \quad \forall p \in \mathcal{P}$

  $\hookrightarrow$ it means if $\lambda_p > 0$, then ϕ could be increased, so the condition (iii) must be tight so this doesn't happen.

  if $\lambda_p > 0 \Rightarrow f_p = 0$. so if the latency of a path is greater than λ, the flow on the path is zero.

  If the flow is >0, then $\lambda_p = \lambda$, the latency on the path is equal to the minimum (else we could increase ϕ because of the stationarity condition).

This is equivalent to f being equilibrium flow ■

Note that we only used the value of $\phi(f)$ in $\frac{\partial \phi}{\partial f_e} = l_e(f_e)$, so if this condition holds and l_e is convex, this should work in the general case. So we define $\phi(f)$ like this:

$$\phi(f) = \sum_{e\in E} \int_0^{f_e} l_e(x)\,dx.$$