Exercises for Unit 9

(1.) a) What is the prob. that you end up with the most valuable gem?

strategy = transfer the first K, remember the largest value M that you saw and then pick the first one whose value $> M$

$S :=$ event that we succeed in picking the most valuable gem

$S_i :=$ event that we succeed when the best one is the i-th one we see

$\hookrightarrow$ S_i's are disjoint $\Rightarrow Pr(S) = \sum_{i=1}^{n} Pr(S_i) = \underbrace{\sum_{i=K+1}^{n} Pr(S_i)}$

$\hookrightarrow$ because we discard the first K

- for S_i to happen, two independent events have to happen:

1. $A_i =$ the most valuable gem is at position i

   $Pr(A_i) = \frac{1}{n}$ because the order in which we are picking them is random

2. $B_i =$ we must not take $K+1$-th, ..., $i-1$-th gem

   $\hookrightarrow$ this happens when the most valuable one among first $i-1$ is in the first K

   $Pr(B_i) = \frac{K}{i-1}$

$$Pr(S) = \sum_{i=K+1}^{n} Pr(A_i) \cdot Pr(B_i) = \sum_{i=K+1}^{n} \frac{1}{n} \cdot \frac{K}{i-1} = \frac{K}{n} \underbrace{\sum_{i=K}^{n-1} \frac{1}{i}}_{\approx \ln n - \ln K} \approx \frac{K}{n}(\ln n - \ln K)$$

b) K s.t. $Pr(S)$ maximized?

$$\frac{d}{dK} Pr(S) = \frac{1}{n}(\ln n - \ln K) + \frac{K}{n} \cdot \left(\frac{-1}{K}\right) = \frac{1}{n}\left(\ln \frac{n}{K} - 1\right) = 0$$

$$\Rightarrow \ln \frac{n}{K} = 1 \Rightarrow \frac{n}{K} = e \Rightarrow K = \frac{n}{e}$$

and with this choice of K, $Pr(S) = \frac{1}{e}$

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(2.) N = number of throws I have to make to outperform the first throw

D_i = distance of the i-th throw

S_t = event that I outperform the 1st throw in the t-th attempt and in no attempt before that

S_t happens exactly when the following conditions hold:

1. D_t is the min of $\{D_1, \dots, D_t\}$
2. D_1 is the min of $\{D_1, \dots, D_{t-1}\}$

$\Rightarrow D_t \leq D_1 \leq D_i \quad , \forall i \in \{2, \dots, t-1\}$

Since all D_i's are independent, any order of the D_i's is equally likely:

$$P(S_t) = P(D_t \leq D_1 < D_i \; , \forall \, i \in \{2, \dots, t-1\}) = \frac{1}{t \cdot (t-1)}$$

$$\Rightarrow E[N] = \sum_{t \geq 2} t \cdot P[S_t] = \sum_{t \geq 2} t \cdot \frac{1}{t(t-1)} = \sum_{t \geq 1} \frac{1}{t}$$

$\Rightarrow$ the expected number of throws is unbounded