Algorithms and Data Structures (WS15/16)

Example Solutions for Units 37 & 38

Problem 1

A simple adjustment of the DA algorithm does the trick. We change the while loop to end when each man has run out of non-forbidden women to propose to. The remaining arguments stay almost exactly the same.

Problem 2

Each edge in $M \cap M'$ matches the same agents in both matchings. Hence, consider $M \oplus M' = (M \cup M') \setminus (M \cap M')$. This set is composed of alternating paths and alternating cycles. For an alternating cycle we again match the same set of agents.

For contradiction, suppose there is an alternating path, which starts, w.l.o.g., with a pair $(x_0, y_0) \in M$, continues with $(x_{i+1}, y_i) \in M'$, and $(x_i, y_i) \in M$, and so on, where $i = 0, 1, \ldots$. Since M' is stable matching, it must be $x_1 \succ_{y_0} x_0$. Otherwise (x_0, y_0) would be blocking pair in M', since x_0 is unmatched in M'. For similar reasons, since M' is stable matching and has no blocking pair, it must be $y_1 \succ_{x_1} y_0$. We continue to argue in this fashion until we reach the end of the path. If final agent is some y_i, then it is single in M', so (x_i, y_i) is a blocking pair in M'. Similarly, if the final agent is some x_i, then (x_i, y_{i-1}) is blocking pair in M. This contradicts that both M and M' are stable matchings.

Problem 3

Each edge in $M \cap M^*$ counts for both matchings. Hence, again consider $M \oplus M^* = (M \cup M^*) \setminus (M \cap M^*)$. If there is an alternating path without an edge of M, then it is a single edge of M^*. This means M is not stable, since the edge is a blocking pair for M. Otherwise, each path or cycle with $k \geq 1$ edges from M has at most $\ell \leq k + 1$ edges from M^*. Thus, for each such path or cycle it holds $k \geq \ell/2$. By summing over all paths and cycles, the inequality continues to hold for the overall cardinality of the matchings.

Problem 4

Consider three men $\mathcal{X} = \{A, B, C\}$ and three women $\mathcal{Y} = \{1, 2, 3\}$. The preferences are

A	$2 \succ_A 1 \succ_A 3$
B	$1 \succ_B 3 \succ_B 2$
C	$1 \succ_C 2 \succ_C 3$
1	$A \succ_1 C \succ_1 B$
2	$C \succ_2 A \succ_2 B$
3	$A \succ_3 C \succ_3 B$

If we run the DA algorithm, the stable matching is $(A, 2), (B, 3), (C, 1)$. So woman 1 gets man C. If 1 switches to report $A \succ_1 B \succ_1 C$, then she rejects C for B, which means C then proposes to woman 2, who rejects A, and A proposes to 1. The resulting matching is $(A, 1), (B, 3), (C, 2)$, which gives a better partner for 1.

Problem 5

Consider the example from the lecture with two men $\mathcal{X} = \{A, B\}$ and two women $\mathcal{Y} = \{1, 2\}$ with preferences $1 \succ_A 2$, $2 \succ_B 1$, $B \succ_1 A$ and $A \succ_2 B$. Now start with the empty matching $M = \emptyset$ and consider the following cycle of blocking pair resolutions: $(A, 1), (B, 1), (B, 2), (A, 2), (A, 1)$.