Algorithms and Data Structures (WS15/16)

Example Solutions for Unit 4

Problem 1

A simple algorithm to solve this problem is sorting both sets A and B and then doing a linear scan to check for common elements. We start from the smallest element in both sets: a_1 and b_1. At each step, if the two elements are equal, we return **false**. Otherwise, if a_i is smaller, then we move forward in A, else we move forward in B. When we reach the end of one of the sets, it means the sets are disjoint and we return **true**.

```
procedure Disjointness(A, B)
    Sort (A)
    Sort (B)
    i ← 1
    j ← 1
    while i ≤ n and j ≤ n do
        if A[i] = B[j] then return false
        else if A[i] < B[j] then
            i ← i + 1
        else
            j ← j + 1
    return true
```

The algorithm takes $O(n \log n)$ for sorting and $O(n)$ for scanning, therefore in total $O(n \log n)$.

Now we prove that even in the algebraic computation model, any algorithm has to take time $\Omega(n \log n)$.

Consider the following set:

$$W = \left\{(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n) \in \mathbb{R}^{2n} : a_i \neq b_j \; \forall 1 \leq i, j \leq n\right\}$$

Then the problem DISJOINTNESS for A and B is equivalent to testing membership of $(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$ in W. In order to bound the running time, we need to look at the number of connected components of W.

Let σ be a permutation of A. We claim that all sets of the following type are in distinct connected components of W:

$$S_\sigma = \left\{(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n) \in \mathbb{R}^{2n} : a_{\sigma_1} < b_1 < a_{\sigma_2} < b_2 < \cdots < a_{\sigma_n} < b_n\right\}$$

Clearly, $S_\sigma \in W$. Any two such sets are separated by a hyperplane where $a_i = b_j$ for some $1 \leq i, j \leq n$, therefore these sets are in distinct connected components. Since there are $n!$ sets S_σ (one for each permutation), the number of connected components is at least $n!$.

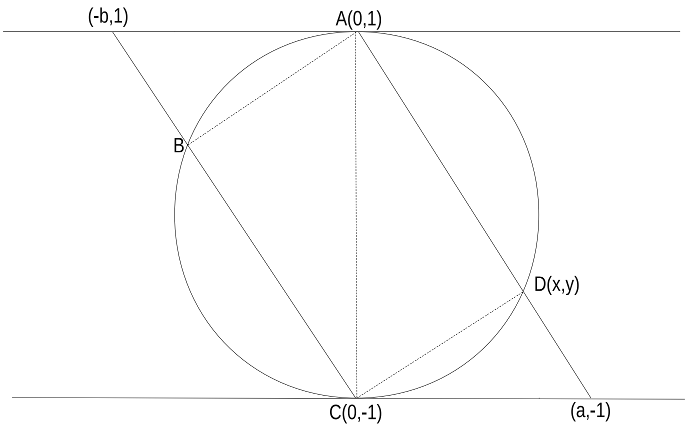
Using the theorem Ben-Or, any algorithm that solves this problem has to take time $\Omega(\log n! - 2n) = \Omega(n \log n)$.

Note that we have not counted all the connected components of W. In particular, if we also consider all permutations of B and also start the interleaving with elements of B, we can get in total at least $2(n!)^2$ connected components, but this does not improve the lower bound.

Problem 2

a)

The transformation from DISJOINTNESS to DIAMETER uses operations sin and cos, which are not included in the algebraic computation tree model. Therefore we cannot deduce any lower bound for DIAMETER in this model.



b)

We find another transformation that only uses operations allowed in our model. W.l.o.g we assume that all reals in A and B are positive (if they are not, we can simply shift them by an appropiate amount).

We consider the unit circle and the two lines parallel to the x-axis that intersect the circle in exactly one point (see figure). For each $a \in A$ we do the following construction:

- consider the point on the lower line with coordinates $(a, -1)$
- draw a line passing through this point and point A with coordinates $(0, 1)$
- consider the intersection of this line with the circle (point D in the figure). This will be our p_a

We do a similar construction for all $b \in B$ and obtain q_b (B in figure).

All points in A will be mapped to the right side and all points in B will be mapped to the left side.

Let $P_{AB} = \{p_a : a \in A\} \cup \{q_b : b \in B\}$

We claim that DIAMETER for P_{AB} is equivalent DISJOINTNESS for A, B:

Clearly $a = b$ if and only if p_a and q_b are diametrically opposed (in particular the arc $AB =$ the arc CD). Also we cannot have p_{a_i} diametrically opposed to p_{a_j} since they are both on the same side of the circle. Similarly for any q_{b_i} and q_{b_j}.

We only have to show that this transformation fits our model. For that we compute the coordinates (x, y) of point p_a (D in the figure). Since it is on the unit circle we have:

$$x^2 + y^2 = 1$$

Since it is on the line passing through point A and point $(a, -1)$ we can say:

$$\frac{x-0}{a-0} = \frac{y-1}{-2}$$

There are two points satisfying these two equations: A and D. We discard the solution corresponding to A and obtain that:

$$y = \frac{a^2-4}{a^2+4}$$

$$x = \frac{4a}{a^2+4}$$

In a similar manner the coordinates of q_b are:

$$y_b = \frac{-b^2+4}{b^2+4}$$

$$x_b = \frac{-4b}{b^2+4}$$

Therefore our transformation is rational and it fixes the argument in part a).