Algorithms and Data Structures
Spring 2018
Exercises for Units 6 and 7

1. Compute the square of the following polynomial using the method of Karatsuba-Ofman discussed in class.
$$p(x) \quad = \quad 2x^3 + 4x^2 + x + 3$$

2. Assume you have a program Π at your disposal that can square any polynomial of degree less than n in time $T(n)$. Show how to use Π so that you can multiply two polynomials of degree less than n in time $O(T(n))$.

3. Let $T(n)$ be the time necessary to multiply two polynomials of degree less than n. In class we showed that $T(n) = O(n^{\log(2b-1)/\log b})$ for any integer $b > 1$ and thus by choosing b appropriately large $T(n) = O(n^{1+\varepsilon})$ for any $\varepsilon > 0$.

   Show that $T(2^{\binom{k}{2}}) \leq C \cdot k \cdot 2^{\binom{k+1}{2}}$ for some fixed constant $C > 0$, and all integers $k > 1$.

   Hint: make the divide factor b dependent on k.

4. Consider the problem of multiplying two n digit integers. Can polynomial multiplication be of help? What are the relationships between the running times of the algorithms?

5. Compute the square of the following polynomial using the Fourier transform method discussed in class.
$$p(x) \quad = \quad 2x^3 + 4x^2 + x + 3$$
   Use $\omega = (1+i)/\sqrt{2}$ as primitive 8th root of unity, where $i = \sqrt{-1}$.

6. An $N \times N$ matrix A is called a *Toeplitz matrix* iff you have $A_{i,j} = A_{i-1,j-1}$ for all $1 < i, j \leq N$.
   (a) Give an explicit, non-trivial example of a 4×4 Toeplitz matrix
   (b) How do you minimally specify an $N \times N$ Toeplitz matrix?
   (c) Show that the product of an $N \times N$ Toeplitz matrix A and an N-vector b can be computed in $O(N \log N)$ time.