1. Consider the following scaling algorithm for finding the maximum flow in a flow network
\( G = (V, E; s, t; c) \) where all the capacities are integral and \( C \) is the maximum capacity.

\[
\text{Max-Flow-By-Scaling}(G) \\
1 \quad \text{let } C \text{ be the maximum capacity} \\
2 \quad f = 0 \\
3 \quad K = 2^{\lceil \log_2 C \rceil} \\
4 \quad \textbf{while } K \geq 1 \textbf{ do} \\
5 \quad \quad \textbf{while } \exists \text{ an augmenting path } \pi \text{ of capacity at least } K \textbf{ do} \\
6 \quad \quad \quad \text{augment flow } f \text{ along } \pi \\
7 \quad \quad K = K/2 \\
8 \quad \text{return } f
\]

(a) Argue that \text{Max-Flow-By-Scaling} returns a maximum flow.
(b) Show that the residual capacity of some cut of \( G \) is at most \( 2K|E| \) each time line 4 is executed.
(c) Argue that the loop in lines 5–6 is executed \( O(|E|) \) times for each value of \( K \).
(d) Show that \text{Max-Flow-By-Scaling} can be implemented to run in time \( O(|E|^2 \log C) \).

2. Show that if all the capacities of a network are 1, then the maximum flow problem can be solved in time \( O(n^{2/3}(n + m)) \) or alternatively in time \( O(m^{1/2}(n + m)) \).

3. A vertex cover \( A \) is a subset of the vertices of a graph \( G = (V, E) \) such that for all edges at least one endpoint is in \( A \).

Prove the following Theorem (called König-Egerváry Theorem): In a bipartite graph, the size of a minimum vertex cover equals the size of a maximum matching.

\text{Hint: you may use the connection of the maximum matching problem to maximum flows and minimum cuts.}