Algorithms and Data Structures
Spring 2018
Exercises for Unit 25

1. Consider the following scaling algorithm for finding the maximum flow in a flow network $G = (V, E; s, t; c)$ where all the capacities are integral and C is the maximum capacity.

   MAX-FLOW-BY-SCALING(G)
   1 let C be the maximum capacity
   2 $f = 0$
   3 $K = 2^{\lfloor \log_2 C \rfloor}$
   4 **while** $K \geq 1$ **do**
   5   **while** $\exists$ an augmenting path π of capacity at least K **do**
   6   augment flow f along π
   7     $K = K/2$
   8 return f

   (a) Argue that MAX-FLOW-BY-SCALING returns a maximum flow.

   (b) Show that the residual capacity of some cut of G is at most $2K|E|$ each time line 4 is executed.

   (c) Argue that the loop in lines 5–6 is executed $O(|E|)$ times for each value of K.

   (d) Show that MAX-FLOW-BY-SCALING can be implemented to run in time $O(|E|^2 \log C)$.

2. Show that if all the capacities of a network are 1, then the maximum flow problem can be solved in time $O(n^{2/3}(n + m))$ or alternatively in time $O(m^{1/2}(n + m))$.

3. A **vertex cover** A is a subset of the vertices of a graph $G = (V, E)$ such that for all edges at least one endpoint is in A.

   Prove the following Theorem (called König-Egerváry Theorem): In a bipartite graph, the size of a minimum vertex cover equals the size of a maximum matching.

   Hint: you may use the connection of the maximum matching problem to maximum flows and minimum cuts.