The Union-Find Problem

Divide-and-Conquer Recurrences, Baby Version

Divide-and-Conquer Recurrences, Baby Version

Typical Divide-and-Conquer:

If problem set S has size $n=1$, then nothing to be done.

Otherwise:

* partition S into subproblems of size $< f(n)$
* solve each of the $n/f(n)$ subproblems recursively
* combine subsolutions.

Divide-and-Conquer Recurrences, Baby Version

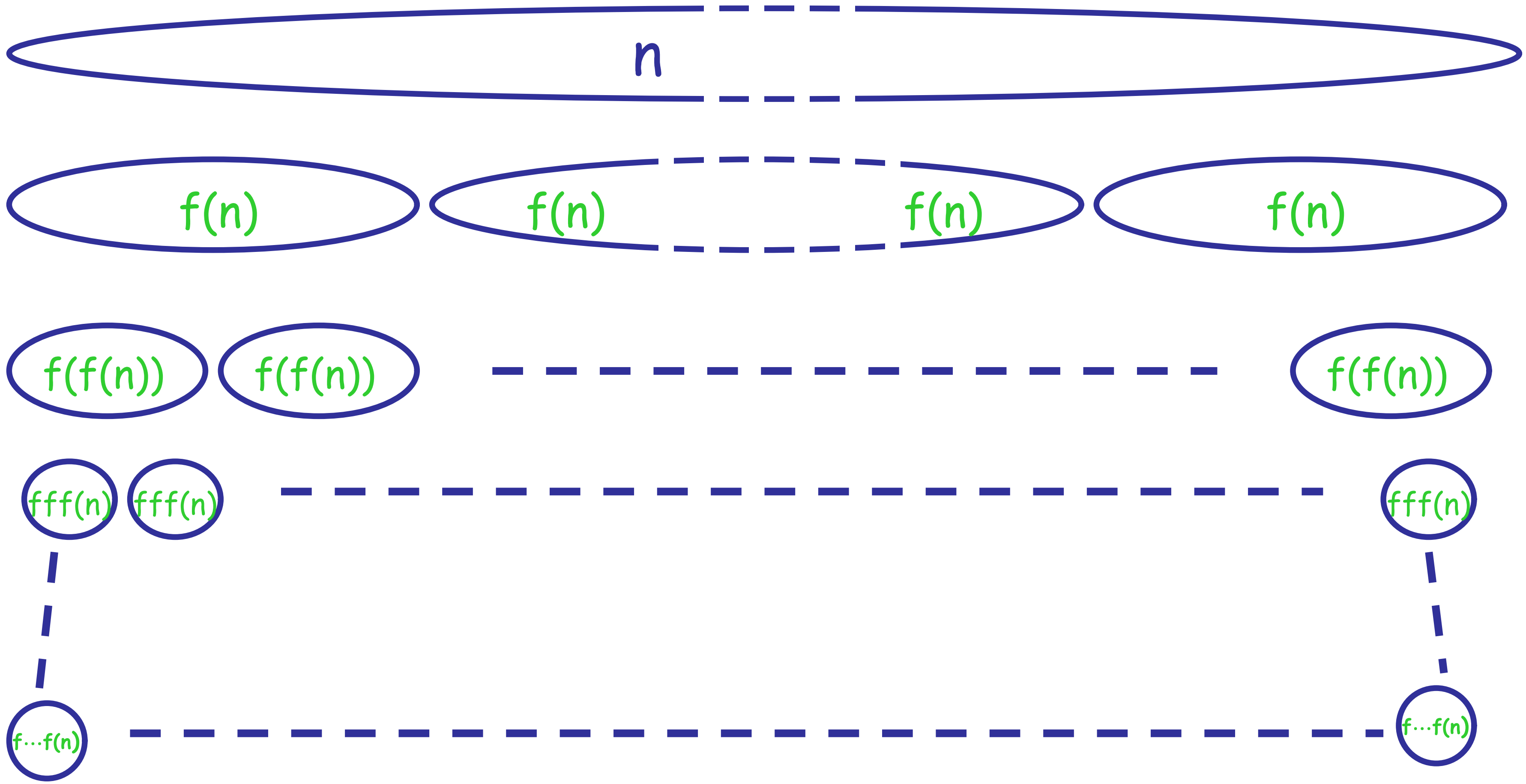
Typical Divide-and-Conquer:

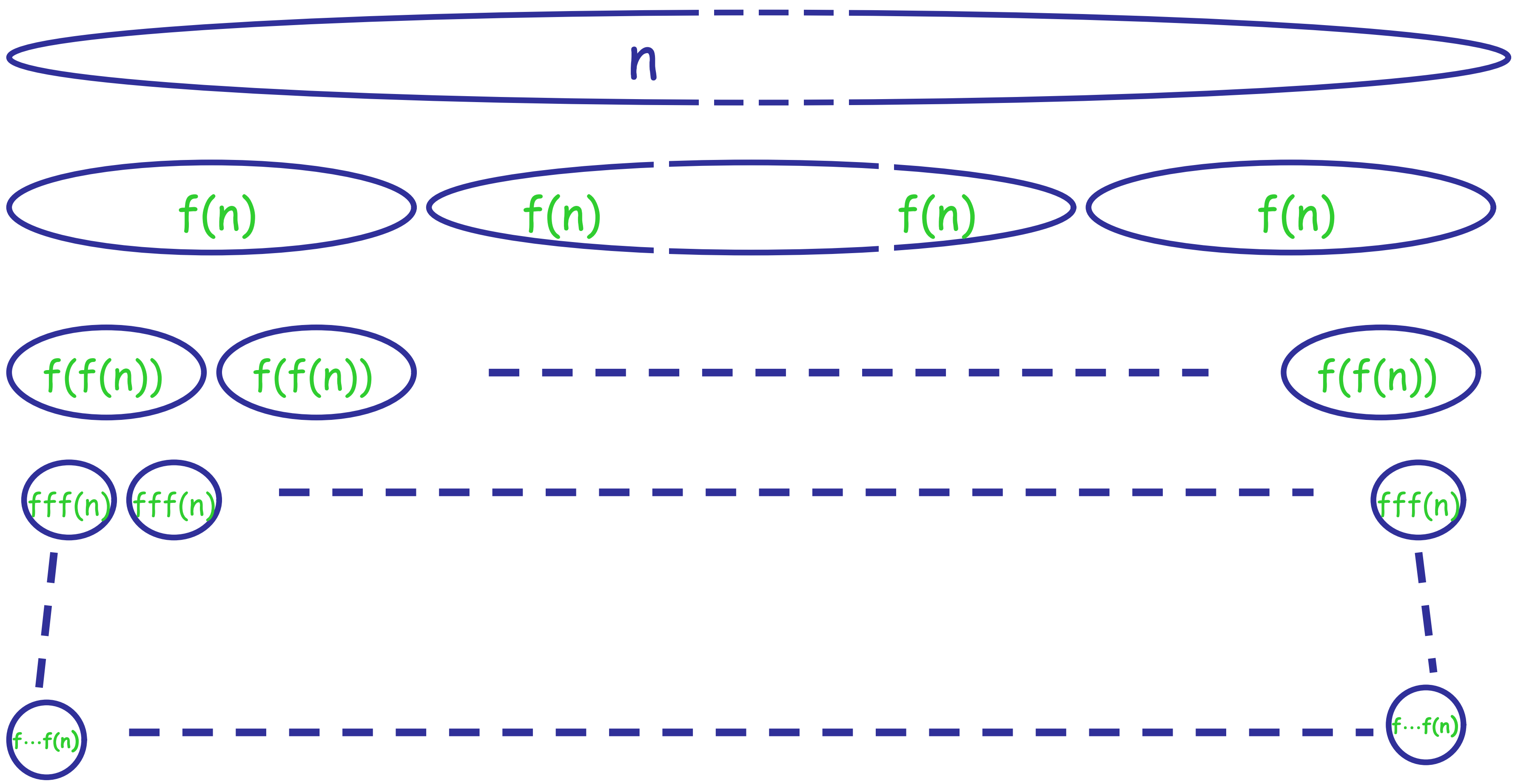
If problem set S has size $n=1$, then nothing to be done.

Otherwise:

* partition S into subproblems of size $< f(n)$
* solve each of the $n/f(n)$ subproblems recursively
* combine subsolutions.

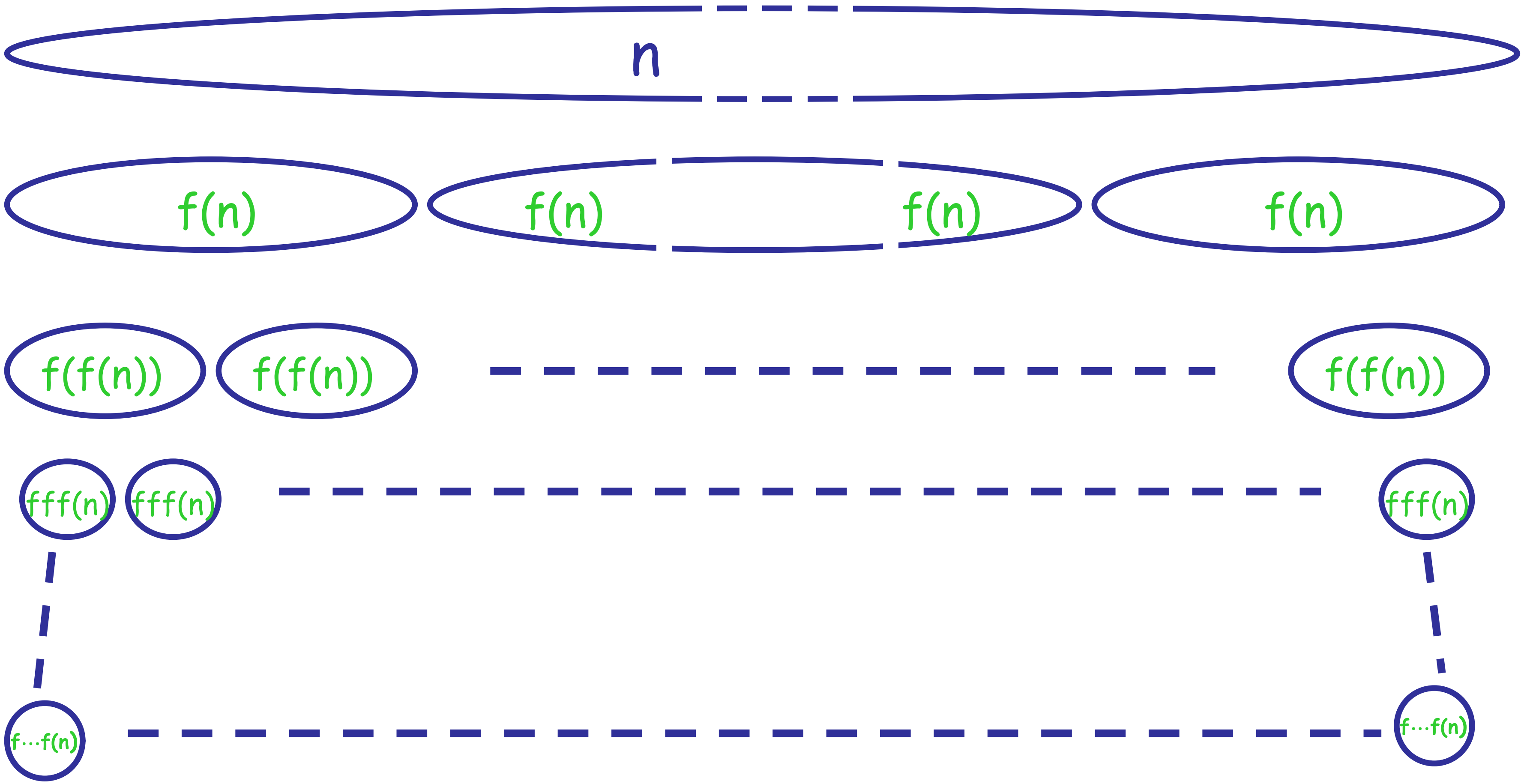
(f needs to satisfy contraction condition $f(n)<n$ for $n>1$.)


n
f(n)
f(n)
f(n)
f(n)
f(f(n))
f(f(n))
f(f(n))
fff(n)
fff(n)
fff(n)
f⋯f(n)
f⋯f(n)



Recurrence:

$$X(n) \le \begin{cases} 0 & \text{if } n \le 1 \\ a \cdot n + \dfrac{n}{f(n)} \cdot X(f(n)) & \text{if } n > 1 \end{cases}$$



Recurrence:

$$X(n) \leq \begin{cases} 0 & \text{if } n \leq 1 \\ a \cdot n + \dfrac{n}{f(n)} \cdot X(f(n)) & \text{if } n > 1 \end{cases}$$

Solution: $X(n) \leq a \cdot n \cdot f^{*}(n)$

$$f^*(n) = \begin{cases} 0 & \text{if } n \leq 1 \\ 1 + f^*(\, f(n)\,) & \text{if } n > 1 \end{cases}$$

$$f^*(n) = \begin{cases} 0 & \text{if } n \leq 1 \\ 1 + f^*(f(n)) & \text{if } n > 1 \end{cases}$$

$$f^*(n) = \min \{ k \mid \underbrace{f(f(\cdots\cdots f}_{k \text{ times}}(n)\cdots) \leq 1 \}$$

$$f^*(n) = \begin{cases} 0 & \text{if } n \le 1 \\ 1 + f^*(\ f(n)\ ) & \text{if } n > 1 \end{cases}$$

$$f^*(n) = \min\{\ k \mid \underbrace{f(f(\ \cdots\cdots\ f}_{k \text{ times}}(n)\cdots) \le 1\ \}$$

Properties:

1) $f^*(\ f(n)\ ) = f^*(n) - 1$

2) f a "nice" compaction
   $\Rightarrow f^*$ a "nice" compaction and
   f^* "much smaller" than f

Examples for f^* :

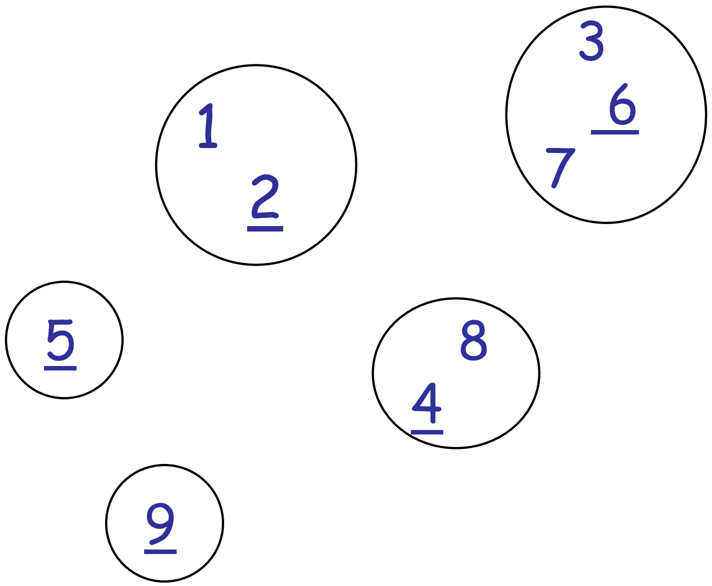
$f(n)$	$f^*(n)$
$n-1$	$n-1$
$n-2$	$n/2$
$n-c$	n/c
$n/2$	$\log_2 n$
n/c	$\log_c n$
$\sqrt{n}$	$\log \log n$
$\log n$	$\log^* n$

Union Find with Path Compressions

Union Find with Path Compressions

Maintain partition of $S = \{1, 2, \cdots, n\}$

under operations

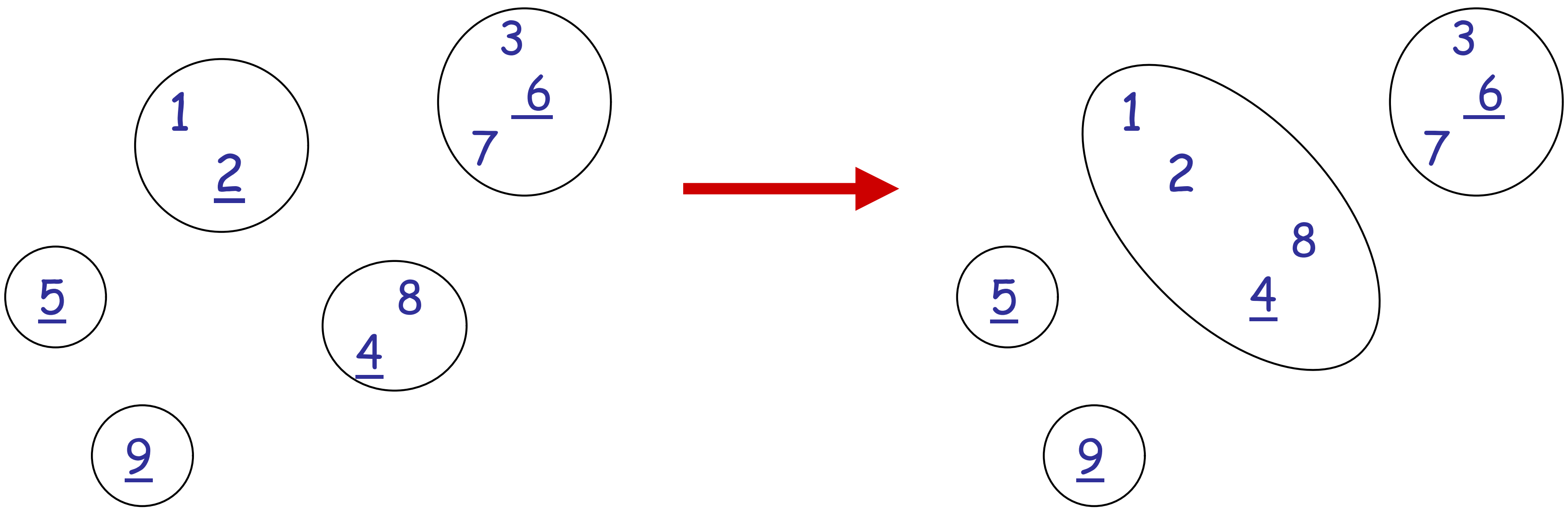


Union Find with Path Compressions

Maintain partition of $S = \{1, 2, \cdots, n\}$

under operations

Union(2, 4)

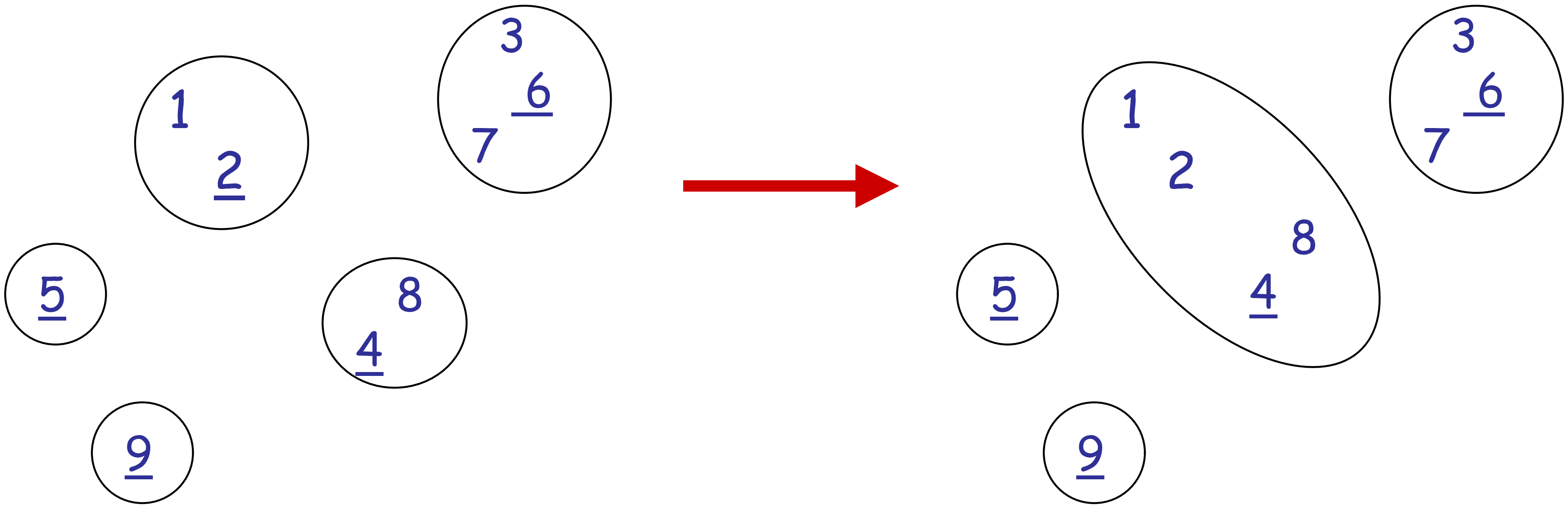


Union Find with Path Compressions

Maintain partition of $S = \{1, 2, \cdots, n\}$

under operations

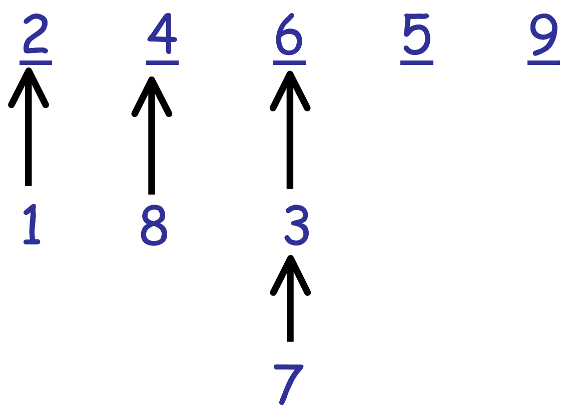
Union(2 , 4)



Find(3) = 6 (representative element)

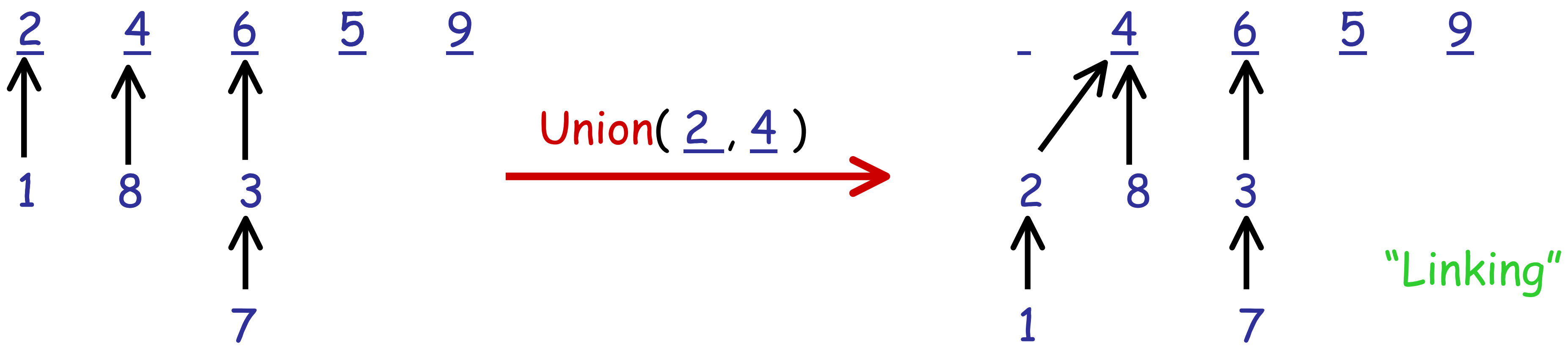
Impementation

* forest $\mathcal{F}$ of rooted trees with node set S
* one tree for each group in current partition
* root of tree is representative of the group



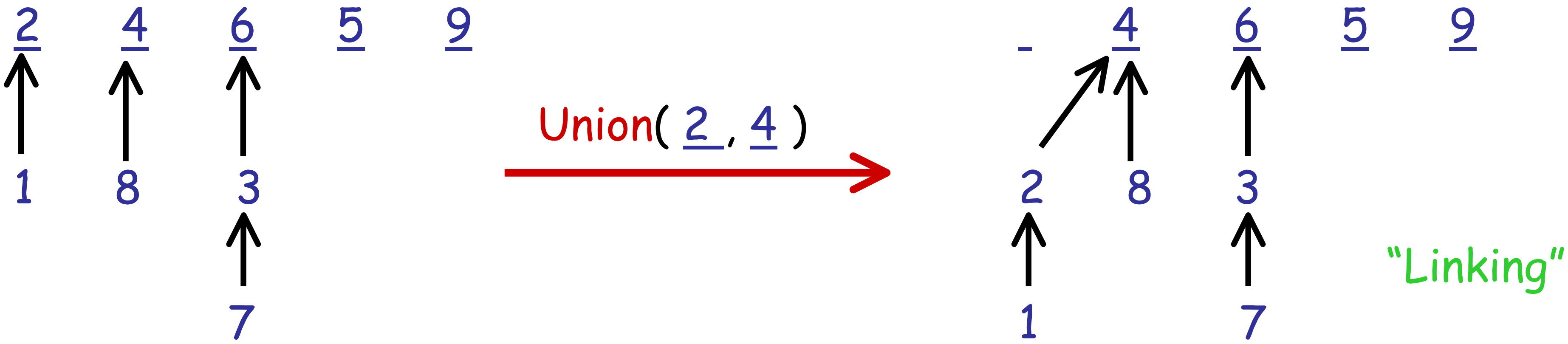
Impementation

* forest $\mathcal{F}$ of rooted trees with node set S
* one tree for each group in current partition
* root of tree is representative of the group



Impementation

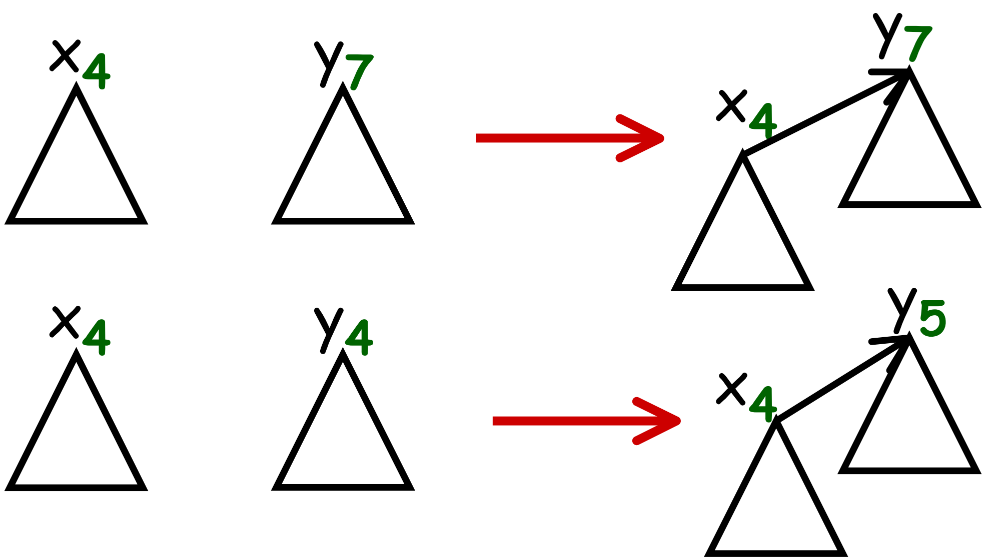
* forest $\mathcal{F}$ of rooted trees with node set S
* one tree for each group in current partition
* root of tree is representative of the group



Find(x)   follow path from x to root   "path follwoing"

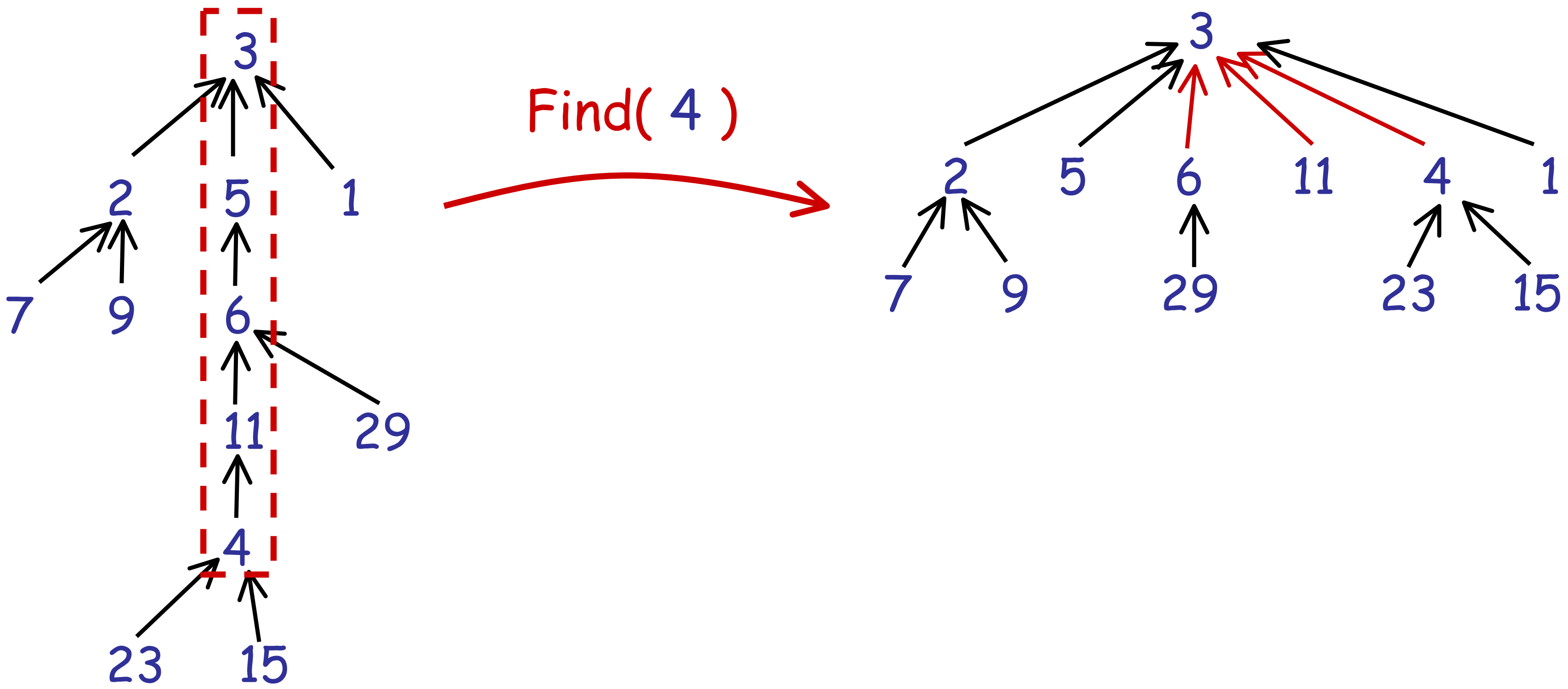
Heuristic 1: "linking by rank"

- each node x carries integer rk(x)
- initially rk(x) = 0
- as soon as x is NOT a root, rk(x) stays unchanged
- for Union(x , y) make node with smaller rank child of the other
  in case of tie, increment one of the ranks



Heuristic 2: Path compression

when performin a Find(x) operation make all nodes in the "findpath" children of the root



sequence of Union and Find operation

Explicit cost model:

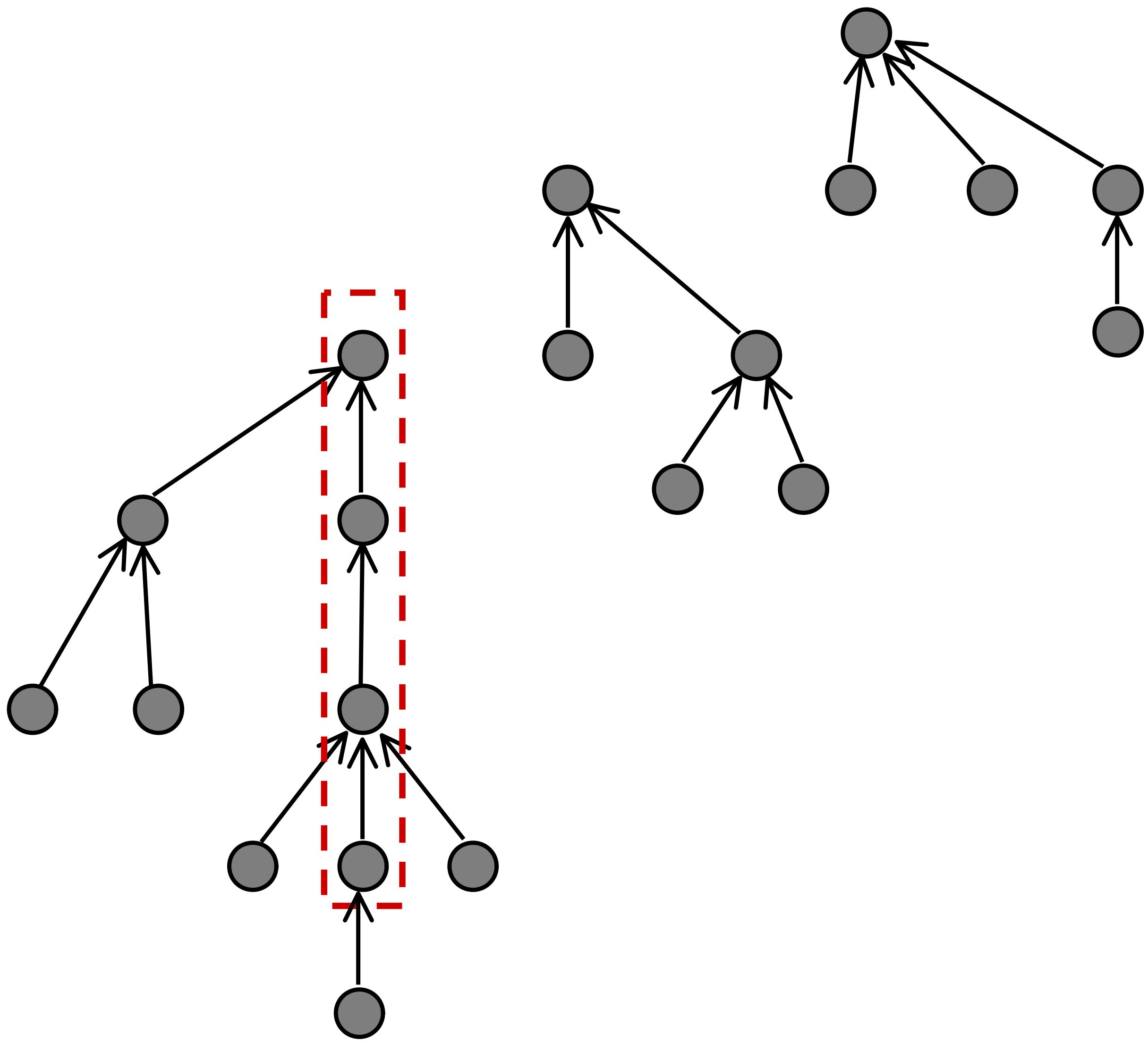
cost(op) = # times some node gets a new parent

Time for Union(x , y) = O(1) = O(cost(Union(x,y)))

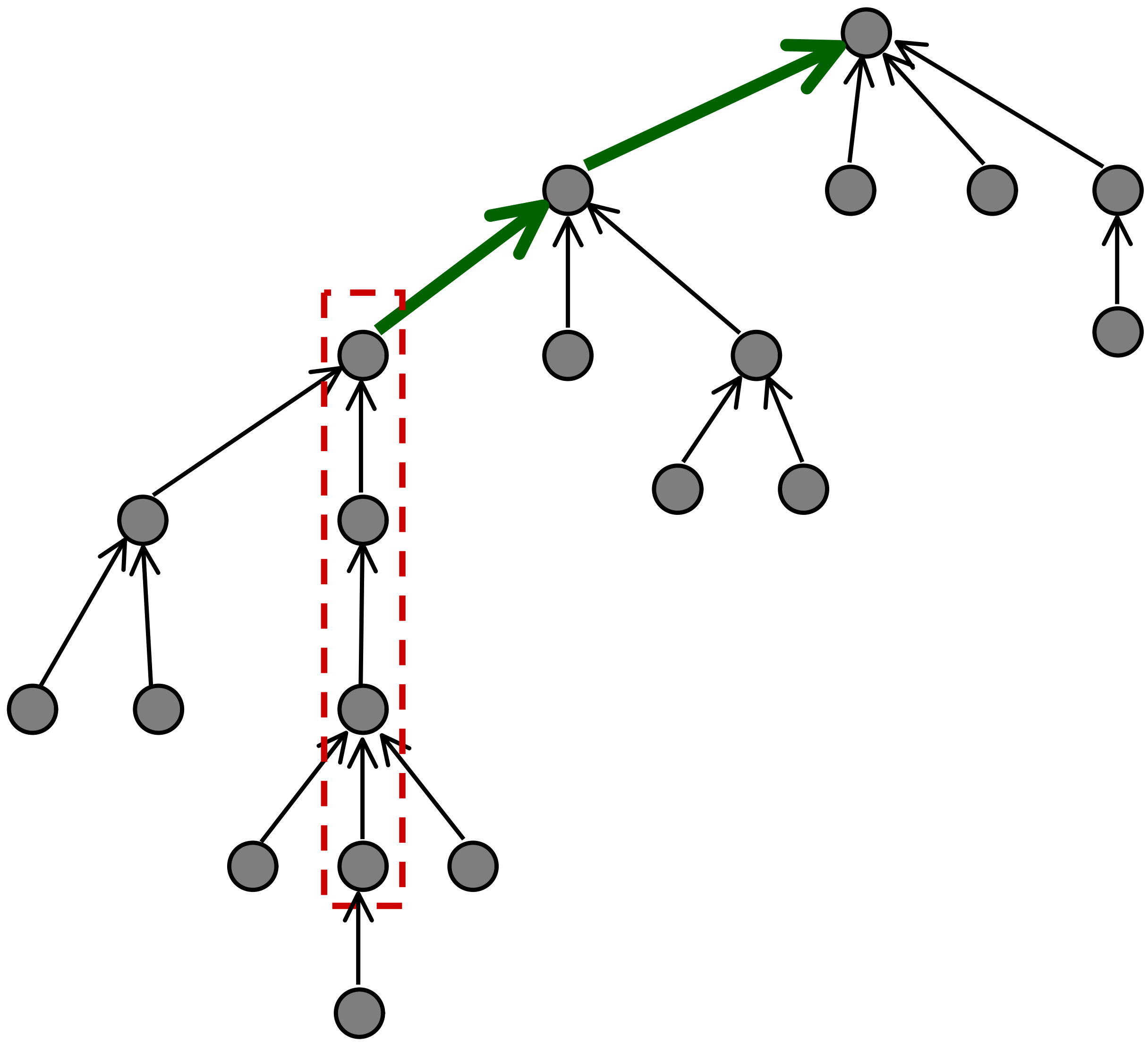
Time for Find(x) = O(# of nodes on findpath)

= O(2 + cost(Find(x)))

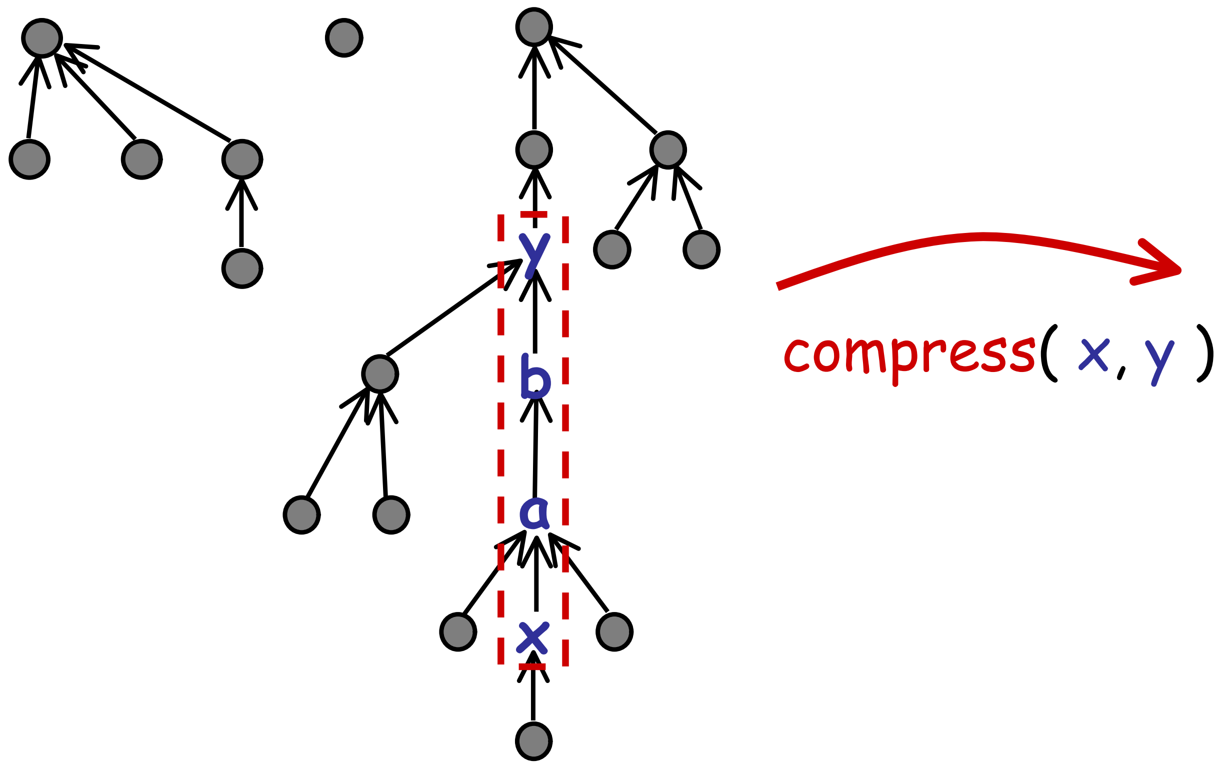
For analysis assume all Unions are performed first, but Find-paths are only followed (and compressed) to correct node.



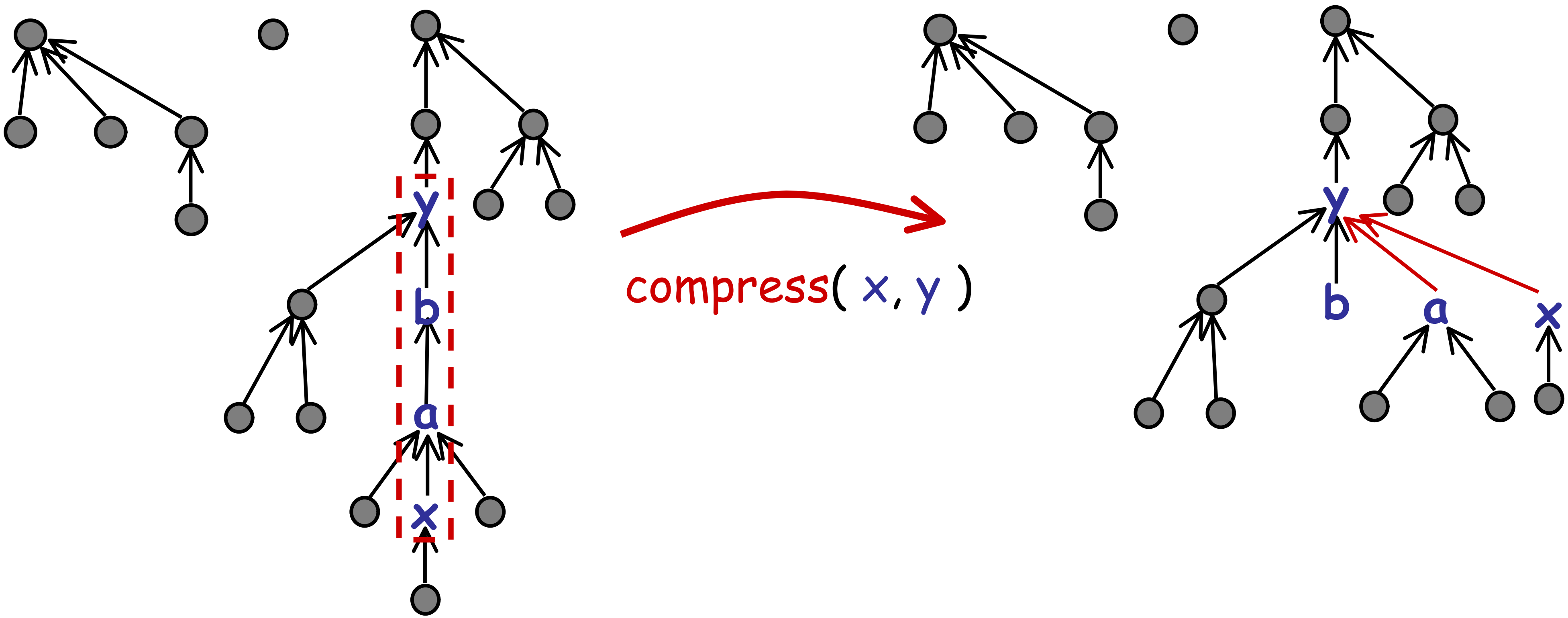
For analysis assume all Unions are performed first, but Find-paths are only followed (and compressed) to correct node.



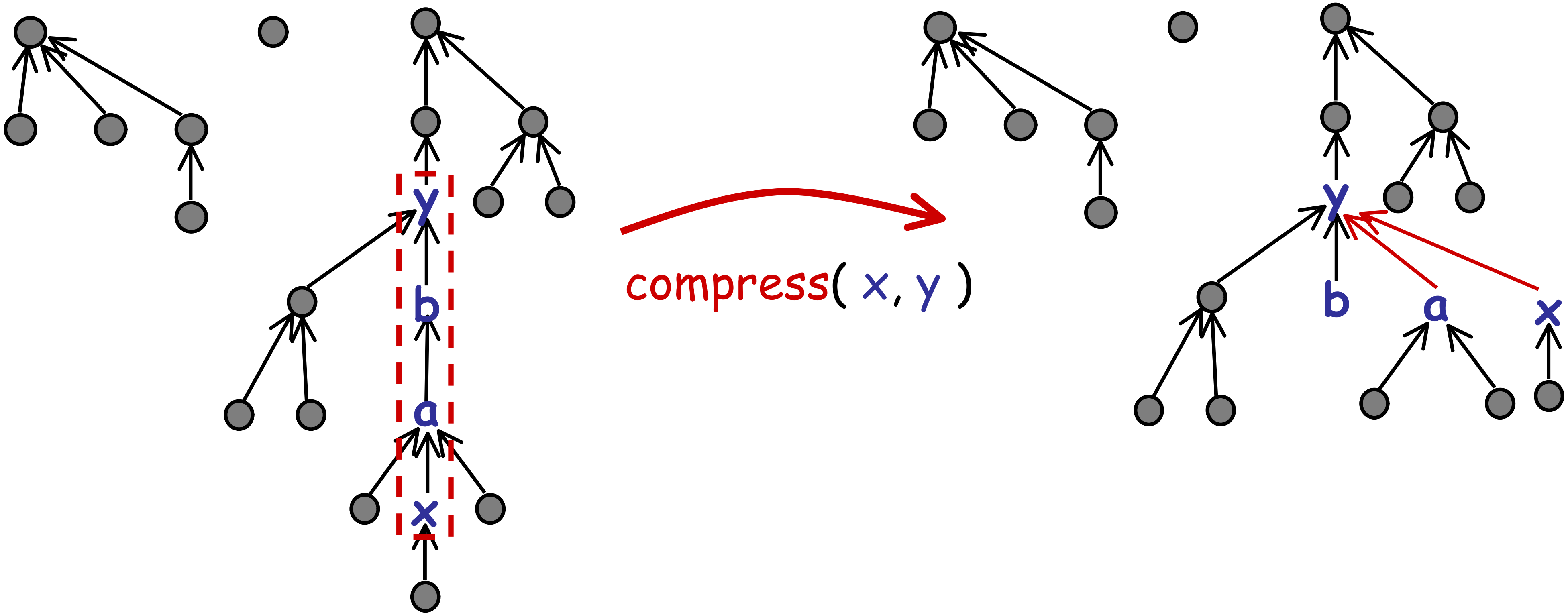
General path compression in forest $\mathcal{F}$



General path compression in forest $\mathcal{F}$



General path compression in forest $\mathcal{F}$



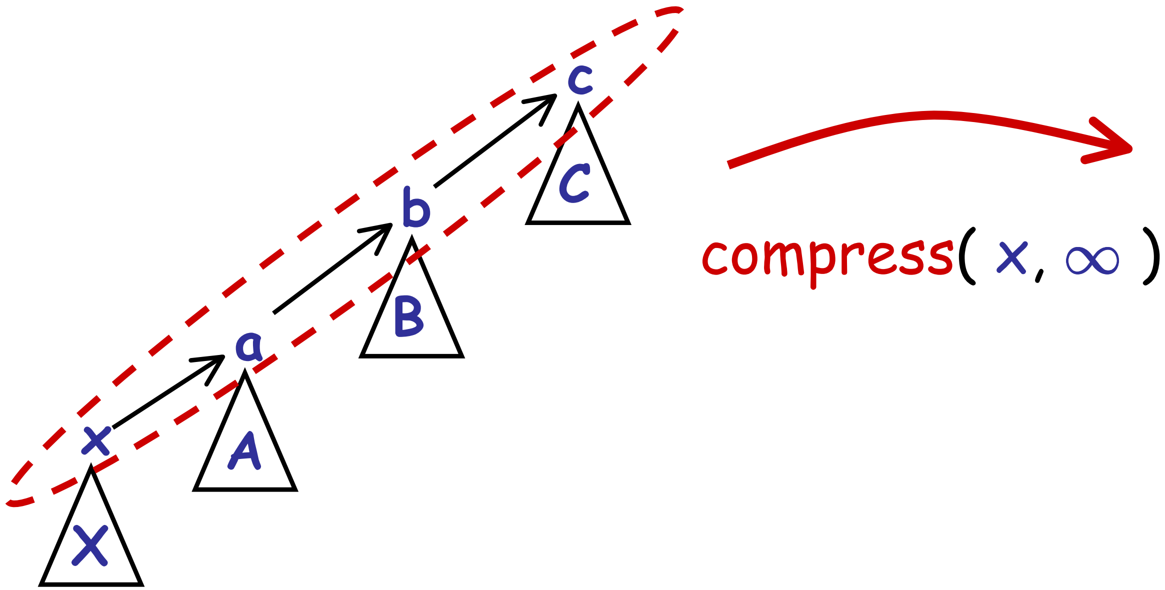
cost(compress(x,y)) = # of nodes that get a new parent

General path compression in forest $\mathcal{F}$

“rootpath compress“

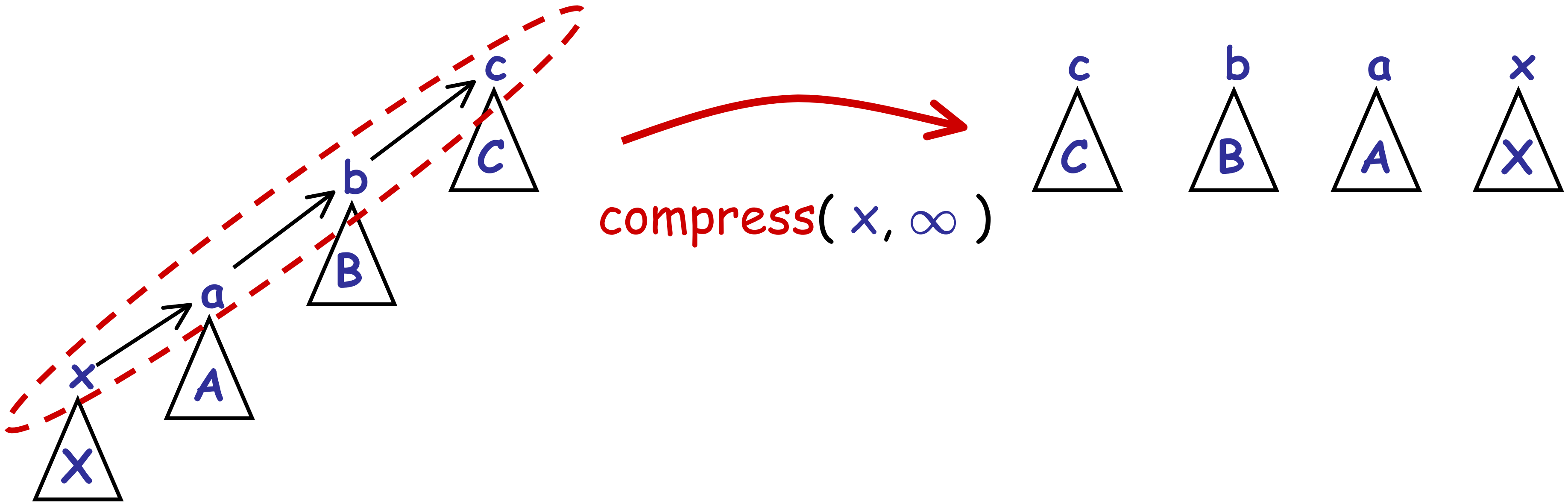
General path compression in forest $\mathcal{F}$

"rootpath compress"



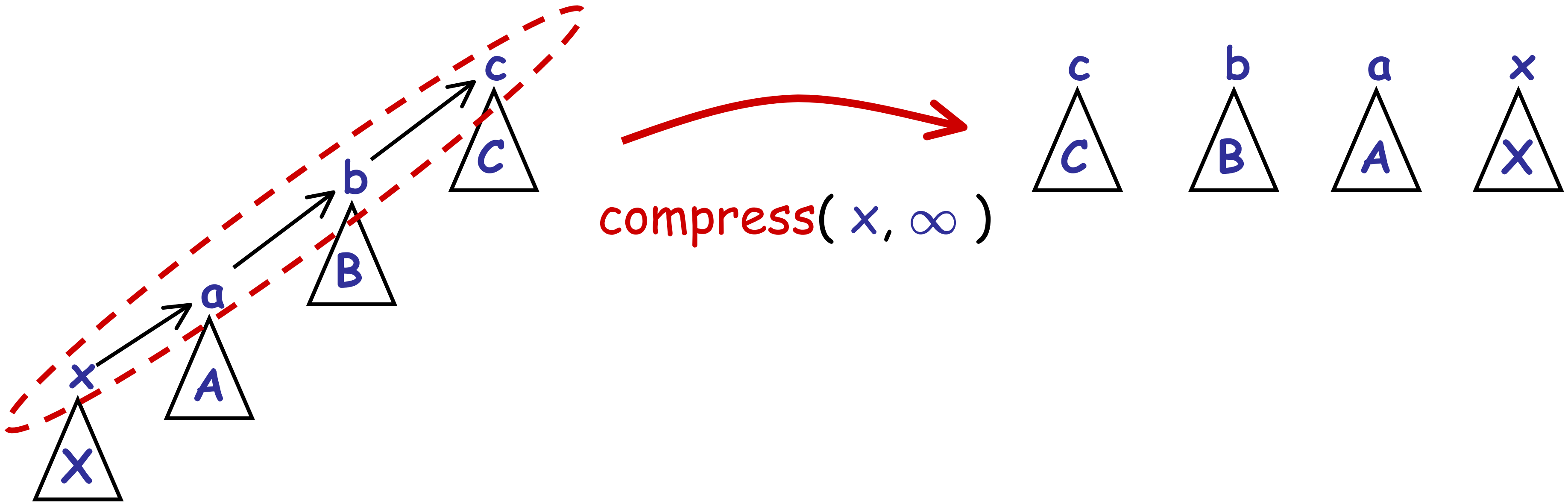
General path compression in forest $\mathcal{F}$

"rootpath compress"



General path compression in forest $\mathcal{F}$

"rootpath compress"



cost(compress(x, ∞)) = # of nodes that get a new parent

= 0

Problem formulation

$\mathcal{F}$ forest on node set X

$\mathcal{C}$ sequence of compress operations on $\mathcal{F}$

$|\mathcal{C}|$ = # of true compress operations in $\mathcal{C}$

(rootpath compresses excluded)

cost($\mathcal{C}$) = $\sum$(cost of individual operations)

Problem formulation

$\mathcal{F}$ forest on node set X

$\mathcal{C}$ sequence of compress operations on $\mathcal{F}$

$|\mathcal{C}|$ = # of true compress operations in $\mathcal{C}$

(rootpath compresses excluded)

cost($\mathcal{C}$) = $\sum$(cost of individual operations)

How large can cost($\mathcal{C}$) be at most,
in terms of |X| and $|\mathcal{C}|$?

Dissection of a forest $\mathcal{F}$ with node set X :

partition of X into "top part" X_t
and "bottom part" X_b

so that top part X_t is "upwards closed",

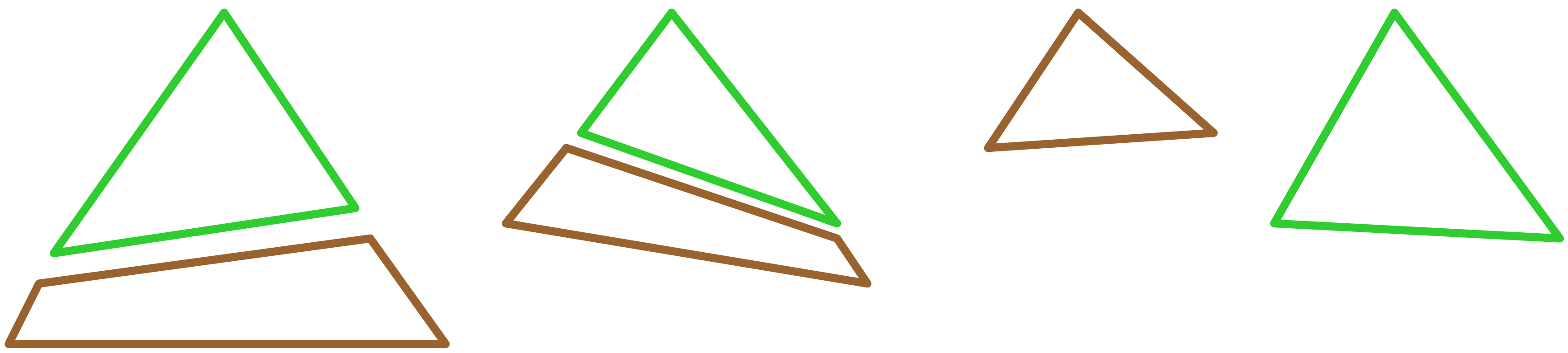
i.e. $x \in X_t \Rightarrow$ every ancestor of x is in X_t also

Dissection of a forest $\mathcal{F}$ with node set X :

partition of X into "top part" X_t
and "bottom part" X_b

so that top part X_t is "upwards closed",

i.e. $x \in X_t \Rightarrow$ every ancestor of x is in X_t also

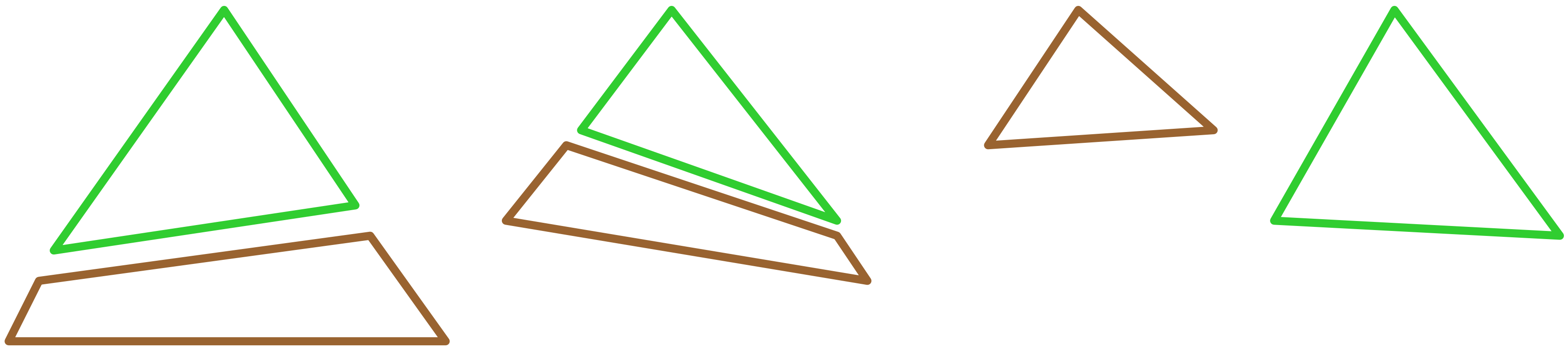


Dissection of a forest $\mathcal{F}$ with node set X :

partition of X into "top part" X_t
and "bottom part" X_b

so that top part X_t is "upwards closed",

i.e. $x \in X_t \Rightarrow$ every ancestor of x is in X_t also



Note: X_t, X_b dissection for $\mathcal{F}$
$\mathcal{F}'$ obtained from $\mathcal{F}$ by
sequence of path compressions } $\Rightarrow$ X_t, X_b is dissection for $\mathcal{F}'$

Main Lemma:

$\mathcal{C}$ ... sequence of operations on $\mathcal{F}$ with node set X

X_t, X_b dissection for $\mathcal{F}$ inducing subforests $\mathcal{F}_t$, $\mathcal{F}_b$

Main Lemma:

$\mathcal{C}$ ... sequence of operations on $\mathcal{F}$ with node set X

X_t, X_b dissection for $\mathcal{F}$ inducing subforests $\mathcal{F}_t$, $\mathcal{F}_b$

$\Rightarrow$ $\exists$ compression sequences
$\mathcal{C}_b$ for $\mathcal{F}_b$ and $\mathcal{C}_t$ for $\mathcal{F}_t$
with

$$|\mathcal{C}_b| + |\mathcal{C}_t| \leq |\mathcal{C}|$$

and

$$\text{cost}(\,\mathcal{C}\,) \leq \text{cost}(\,\mathcal{C}_b\,) + \text{cost}(\,\mathcal{C}_t\,) + |X_b| + |\mathcal{C}_t|$$

Proof: 1) How to get C_b and C_t from C:

Proof: 1) How to get C_b and C_t from C:

compression paths from C

case 1:   into C_t

Proof: 1) How to get C_b and C_t from C:

compression paths from C

case 1: 

into C_t

case 2: 

into C_b

Proof: 1) How to get C_b and C_t from C:

compression paths from C

case 1:

into C_t

case 2:

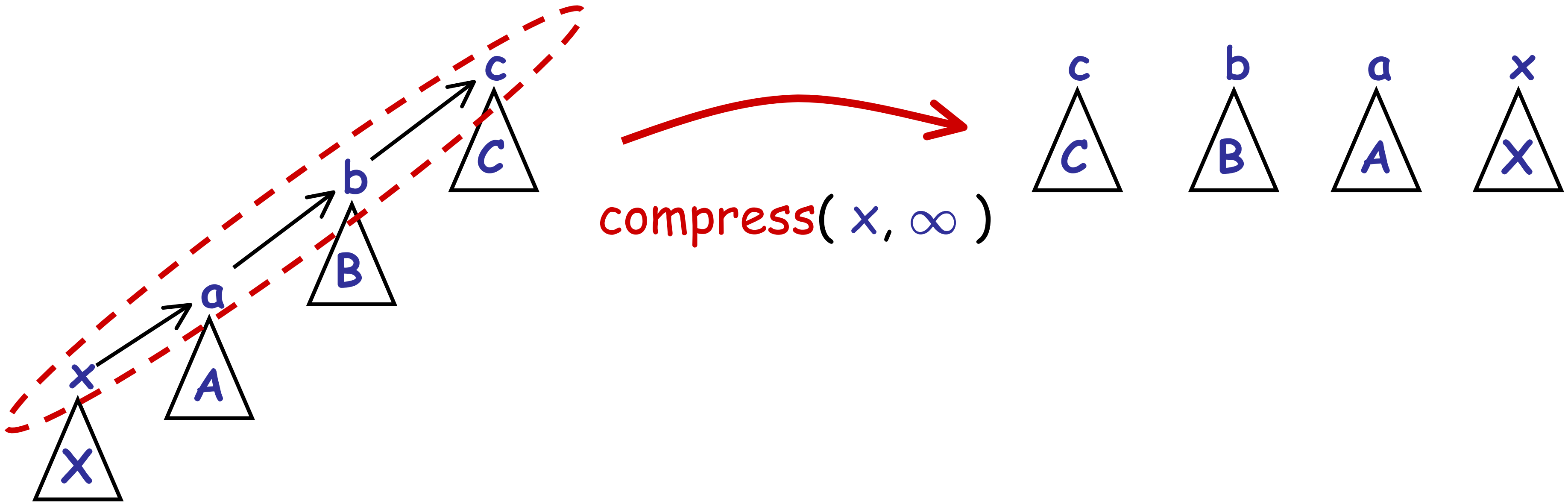
into C_b

case 3:

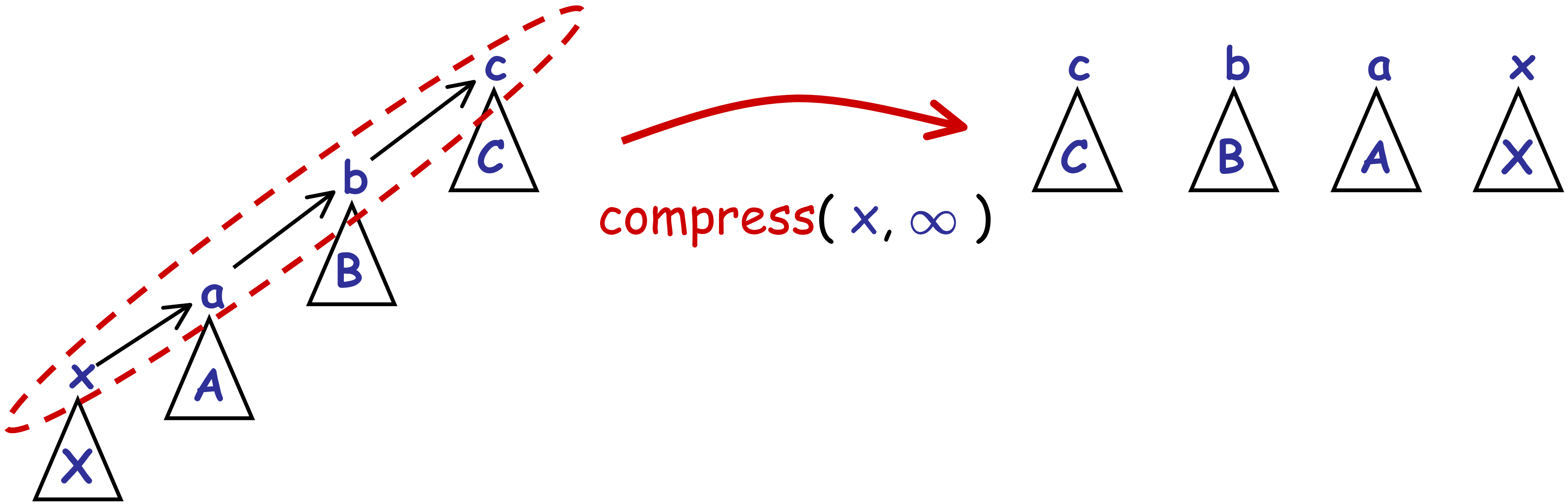
into C_t

into C_b

"rootpath compress"



“rootpath compress”



cost(compress(x, ∞)) = # of nodes that get a new parent

= 0

Proof:

$$|C_b| + |C_t| \leq |C|$$

compression paths from C

case 1:

into C_t

case 2:

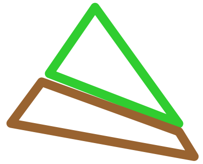
into C_b

case 3:

into C_t

~~into C_b~~

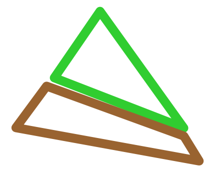
$$\mathrm{cost}(C) \le \mathrm{cost}(C_b) + \mathrm{cost}(C_t) + |X_b| + |C_t|$$



$\mathrm{cost}(C)$

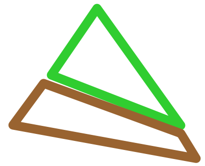
green node gets new green parent:	accounted by $\mathrm{cost}(C_t)$
brown node gets new brown parent:	accounted by $\mathrm{cost}(C_b)$
brown node gets new green parent: for the first time	accounted by $\lvert X_b \rvert$
brown node gets new green parent: again	accounted by $\lvert C_t \rvert$

$$\text{cost}(\mathcal{C}) \leq \text{cost}(\mathcal{C}_b) + \text{cost}(\mathcal{C}_t) + |X_b| + |\mathcal{C}_t|$$



cost($\mathcal{C}$)	
green node gets new green parent:	accounted by cost($\mathcal{C}_t$)
brown node gets new brown parent:	accounted by cost($\mathcal{C}_b$)
brown node gets new green parent: for the first time	accounted by $|X_b|$ - #roots($\mathcal{F}_b$)
brown node gets new green parent: again	accounted by $|\mathcal{C}_t|$

$$\text{cost}(\mathcal{C}) \leq \text{cost}(\mathcal{C}_b) + \text{cost}(\mathcal{C}_t) + |X_b| - \#\text{roots}(\mathcal{F}_b) + |\mathcal{C}_t|$$



$\text{cost}(\mathcal{C})$	
green node gets new green parent:	accounted by $\text{cost}(\mathcal{C}_t)$
brown node gets new brown parent:	accounted by $\text{cost}(\mathcal{C}_b)$
brown node gets new green parent: for the first time	accounted by $|X_b| - \#\text{roots}(\mathcal{F}_b)$
brown node gets new green parent: again	accounted by $|\mathcal{C}_t|$

Main Lemma':

$\mathcal{C}$ ... sequence of operations on $\mathcal{F}$ with node set X

X_t, X_b dissection for $\mathcal{F}$ inducing subforests $\mathcal{F}_t$, $\mathcal{F}_b$

$\Rightarrow$ $\exists$ compression sequences
$\mathcal{C}_b$ for $\mathcal{F}_b$ and $\mathcal{C}_t$ for $\mathcal{F}_t$
with

$$|\mathcal{C}_b| + |\mathcal{C}_t| \leq |\mathcal{C}|$$

and

$$\text{cost}(\mathcal{C}) \leq \text{cost}(\mathcal{C}_b) + \text{cost}(\mathcal{C}_t) + |X_b| - \#\text{roots}(\mathcal{F}_b) + |\mathcal{C}_t|$$

$f(m,n)$ ... maximum cost of any compression sequence C with $|C|=m$ in an arbitrary forest with n nodes.

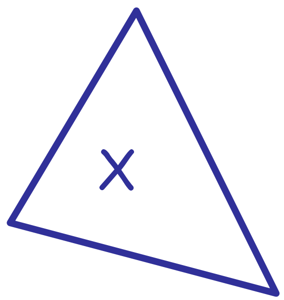
Claim: $f(m,n) \leq (m+n) \cdot \log_2 n$

Claim: $f(m,n) \leq (m+n) \cdot \log_2 n$

Claim: $f(m,n) \leq (m+n) \cdot \log_2 n$

Proof:

forest $\mathcal{F}$



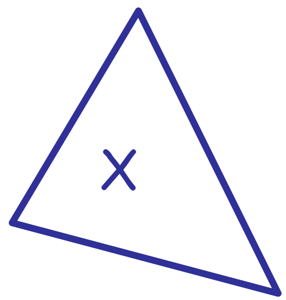
$|X|=n$

C compression sequence $|C|=m$

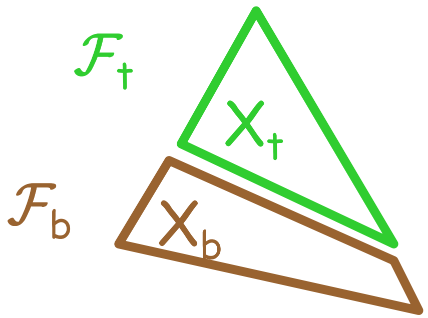
Claim: $f(m,n) \leq (m+n) \cdot \log_2 n$

Proof:

forest $\mathcal{F}$



$|X|=n$



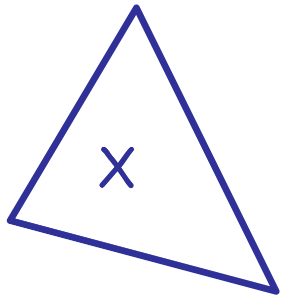
$|X_t|=|X_b|=n/2$

C compression sequence $|C|=m$

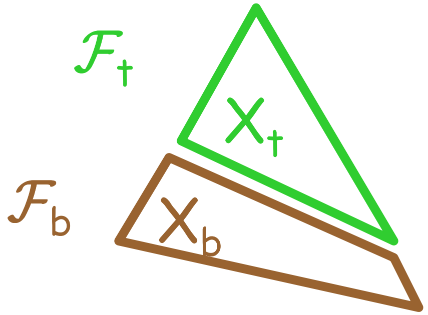
Claim: $f(m,n) \leq (m+n)\cdot\log_2 n$

Proof:

forest $\mathcal{F}$



$|X|=n$



$|X_t|=|X_b|=n/2$

C compression sequence $|C|=m$

Main Lemma $\Rightarrow \exists\ C_t, C_b \quad |C_b|+|C_t| \leq |C|$

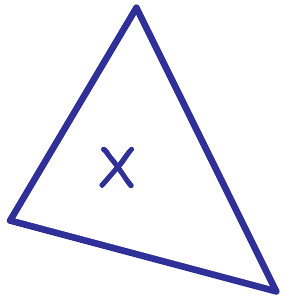
$$m_b + m_t \leq m$$

$$\text{cost}(C) \leq \text{cost}(C_b) + \text{cost}(C_t) + |X_b| + |C_t|$$

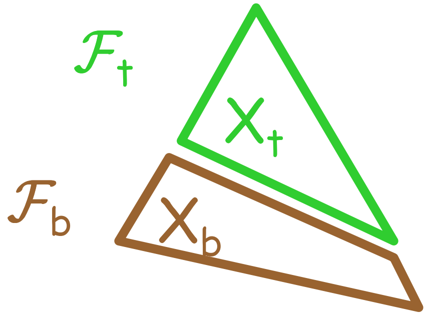
Claim: $f(m,n) \leq (m+n)\cdot\log_2 n$

Proof:

forest $\mathcal{F}$



$|X|=n$



$|X_t|=|X_b|=n/2$

C compression sequence $|C|=m$

Main Lemma $\Rightarrow \exists\ C_t, C_b \quad |C_b|+|C_t| \leq |C|$

$$m_b + m_t \leq m$$

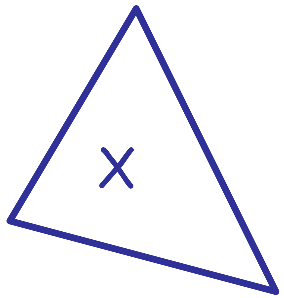
$$\text{cost}(C) \leq \text{cost}(C_b) + \text{cost}(C_t) + |X_b| + |C_t|$$

Induction: $\leq (m_b+n/2)\log n/2 + (m_t+n/2)\log n/2 + n/2 + m_t$

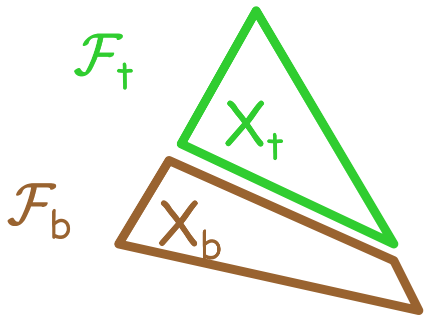
Claim: $f(m,n) \leq (m+n)\cdot\log_2 n$

Proof:

forest $\mathcal{F}$



$|X|=n$



$|X_t|=|X_b|=n/2$

C compression sequence $|C|=m$

Main Lemma $\Rightarrow \exists\, C_t, C_b \quad |C_b|+|C_t| \leq |C|$

$$m_b + m_t \leq m$$

$$\text{cost}(C) \leq \text{cost}(C_b) + \text{cost}(C_t) + |X_b| + |C_t|$$

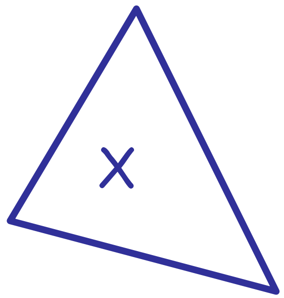
Induction: $\leq (m_b+n/2)\log n/2 + (m_t+n/2)\log n/2 + n/2 + m_t$

$$\leq (m_b+m_t+n/2+n/2)\log n/2 + n + m$$

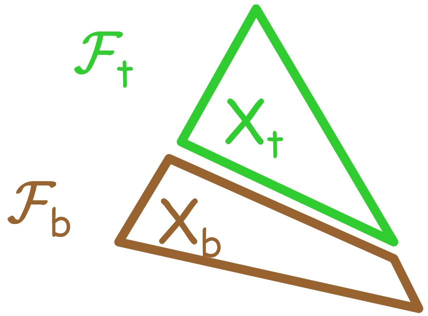
Claim: $f(m,n) \le (m+n)\cdot\log_2 n$

Proof:

forest $\mathcal{F}$



$|X|=n$



$|X_t|=|X_b|=n/2$

C compression sequence $|C|=m$

Main Lemma $\Rightarrow$ $\exists\, C_t, C_b$ $\quad |C_b|+|C_t| \le |C|$

$$m_b + m_t \le m$$

$$\text{cost}(C) \le \text{cost}(C_b) + \text{cost}(C_t) + |X_b| + |C_t|$$

Induction: $\le (m_b+n/2)\log n/2 + (m_t+n/2)\log n/2 + n/2 + m_t$

$$\le (m_b+m_t+n/2+n/2)\log n/2 + n + m$$

$$\le (m+n)\cdot\log_2 n/2 + (m+n) = (m+n)\cdot\log_2 n$$

Corollary:

Any sequence of m Union, Find operations in a universe of n elements that uses arbitrary linking and path compression takes time at most

$$O(\,(m+n)\cdot\log n)$$

Corollary:

Any sequence of m Union, Find operations in a universe of n elements that uses arbitrary linking and path compression takes time at most

$$O(\,(m+n)\cdot \log n)$$

By choosing a dissection that is "unbalanced" in relation to m/n one can prove a better bound of

$$O(\,(m+n)\cdot \log_{\lceil m/n \rceil +1} n)$$

Path compression and union by rank

Path compression and union by rank

Def: $\mathcal{F}$ forest, x node in $\mathcal{F}$

$r(x)$ = height of subtree rooted at x

($r(leaf) = 0$)

$\mathcal{F}$ is a **rank forest**, if

for every node x

for every i with $0 \leq i < r(x)$,

there is a child y_i of x with $r(y_i) = i$.

Path compression and union by rank

Def: $\mathcal{F}$ forest, x node in $\mathcal{F}$

$r(x)$ = height of subtree rooted at x

($r(leaf) = 0$)

$\mathcal{F}$ is a **rank forest**, if

for every node x

for every i with $0 \leq i < r(x)$,

there is a child y_i of x with $r(y_i)=i$.

Note: Union by rank produces rank forests !

Path compression and union by rank

Def: $\mathcal{F}$ forest, x node in $\mathcal{F}$

$r(x)$ = height of subtree rooted at x

($r(\text{leaf}) = 0$)

$\mathcal{F}$ is a **rank forest**, if

for every node x

for every i with $0 \leq i < r(x)$,

there is a child y_i of x with $r(y_i) = i$.

Note: Union by rank produces rank forests !

Lemma: $r(x) = r \Rightarrow x$ has at least r children.

Path compression and union by rank

Def: $\mathcal{F}$ forest, x node in $\mathcal{F}$

$r(x)$ = height of subtree rooted at x

($r(\text{leaf}) = 0$)

$\mathcal{F}$ is a **rank forest**, if

for every node x

for every i with $0 \leq i < r(x)$,

there is a child y_i of x with $r(y_i) = i$.

Note: Union by rank produces rank forests !

Lemma: $r(x)=r \Rightarrow x$ has at least r children and $\geq 2^r$ descendants.

Inheritance Lemma:

$\mathcal{F}$ rank forest with maximum rank r and node set X

$s \in \mathbb{N}$: $X_{>s} = \{ x \in X \mid r(x) > s \}$ $\mathcal{F}_{>s}$

$X_{\leq s} = \{ x \in X \mid r(x) \leq s \}$ $\mathcal{F}_{\leq s}$ induced forests

Inheritance Lemma:

$\mathcal{F}$ rank forest with maximum rank r and node set X

$s \in \mathbb{N}$: $X_{>s} = \{ x \in X \mid r(x) > s \}$ $\quad \mathcal{F}_{>s}$

$X_{\leq s} = \{ x \in X \mid r(x) \leq s \}$ $\quad \mathcal{F}_{\leq s}$ induced forests

i) $X_{\leq s}$, $X_{>s}$ is a dissection for $\mathcal{F}$

ii) $\mathcal{F}_{\leq s}$ is a rank forest with maximum rank $\leq s$

iii) $\mathcal{F}_{>s}$ is a rank forest with maximum rank $\leq r-s-1$

iv) $|X_{>s}| \leq |X| / 2^{s+1}$

Inheritance Lemma:

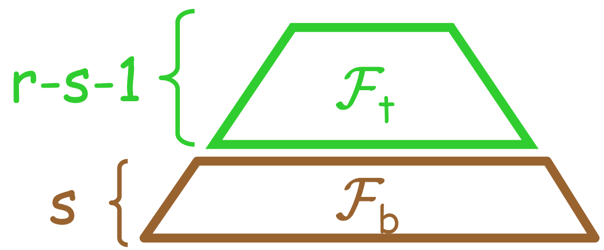
$\mathcal{F}$ rank forest with maximum rank r and node set X

s∈ℕ: $X_{>s} = \{ x \in X \mid r(x) > s \}$ $\mathcal{F}_{>s}$ induced forests

$X_{\leq s} = \{ x \in X \mid r(x) \leq s \}$ $\mathcal{F}_{\leq s}$

i) $X_{\leq s}$, $X_{>s}$ is a dissection for $\mathcal{F}$

ii) $\mathcal{F}_{\leq s}$ is a rank forest with maximum rank $\leq$ s



iii) $\mathcal{F}_{>s}$ is a rank forest with maximum rank $\leq$ r-s-1

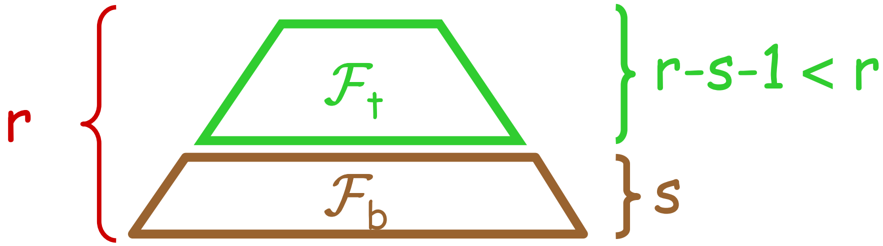
$f(m,n,r)$ = maximum cost of any compression sequence C, with $|C|=m$, in rank forest $\mathcal{F}$ with n nodes and maximum rank r.

$f(m,n,r)$ = maximum cost of any compression sequence C, with $|C|=m$, in rank forest $\mathcal{F}$ with n nodes and maximum rank r.

Trivial bounds:

$$f(m,n,r) \leq (r-1) \cdot n$$

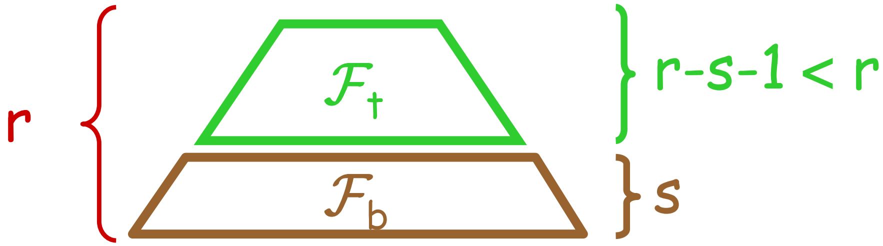
$$f(m,n,r) \leq (r-1) \cdot m$$



$|X_{>s}| = n_t \qquad |C_t| = m_t$

$|X_{\leq s}| = n_b = n - n_t \qquad |C_b| = m_b$

$$\mathrm{cost}(\mathcal{C}) \leq \mathrm{cost}(\mathcal{C}_t) + \mathrm{cost}(\mathcal{C}_b) + |X_b| - \#\mathrm{rts}(\mathcal{F}_b) + |\mathcal{C}_t|$$

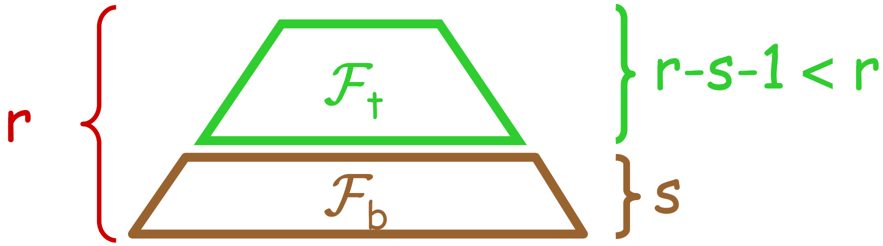


$|X_{>s}| = n_t$ $\quad |C_t| = m_t$

$|X_{\leq s}| = n_b = n - n_t$ $\quad |C_b| = m_b$

$$\text{cost}(\mathcal{C}) \leq \text{cost}(\mathcal{C}_t) + \text{cost}(\mathcal{C}_b) + |X_b| - \#\text{rts}(\mathcal{F}_b) + |\mathcal{C}_t|$$

$$\leq f(m_t, n_t, r-s-1) +$$

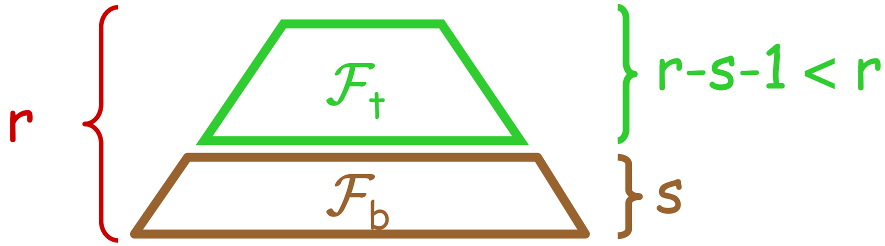


$|X_{>s}| = n_t$ $\quad |C_t| = m_t$

$|X_{\leq s}| = n_b = n - n_t$ $\quad |C_b| = m_b$

$$\mathrm{cost}(\mathcal{C}) \leq \mathrm{cost}(\mathcal{C}_t) + \mathrm{cost}(\mathcal{C}_b) + |X_b| - \#\mathrm{rts}(\mathcal{F}_b) + |\mathcal{C}_t|$$

$$\leq f(m_t, n_t, r-s-1) + f(m_b, n_b, s) +$$

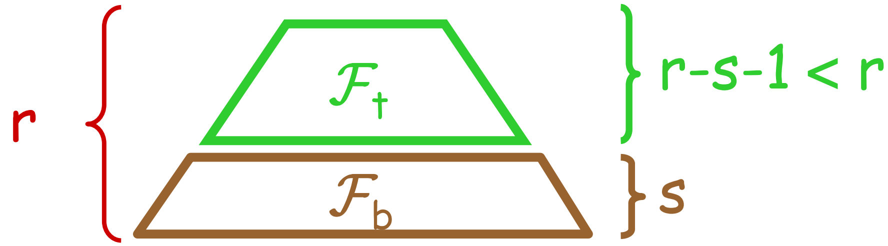


$|X_{>s}| = n_t$ $\quad$ $|C_t| = m_t$

$|X_{\leq s}| = n_b = n - n_t$ $\quad$ $|C_b| = m_b$

$$\text{cost}(\,C\,) \leq \text{cost}(\,C_t\,) + \text{cost}(\,C_b\,) + |X_b| - \#\text{rts}(\mathcal{F}_b) + |C_t|$$

$$\leq f(m_t, n_t, r-s-1) + f(m_b, n_b, s) + n - n_t -$$

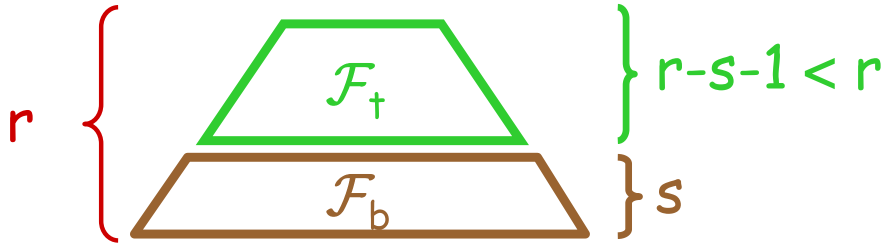


$|X_{>s}| = n_t$ $\quad$ $|\mathcal{C}_t| = m_t$

$|X_{\leq s}| = n_b = n-n_t$ $\quad$ $|\mathcal{C}_b| = m_b$

$$\text{cost}(\,\mathcal{C}\,) \leq \text{cost}(\,\mathcal{C}_t\,) + \text{cost}(\,\mathcal{C}_b\,) + |X_b| - \#\text{rts}(\mathcal{F}_b) + |\mathcal{C}_t|$$

$$\leq f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + n-n_t - (s+1)\cdot n_t +$$



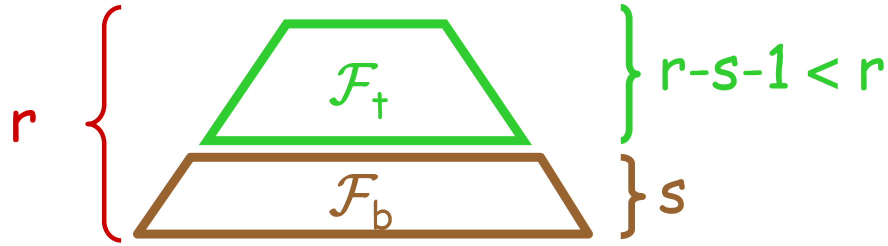
$|X_{>s}| = n_t$ $\quad |C_t| = m_t$

$|X_{\le s}| = n_b = n-n_t$ $\quad |C_b| = m_b$

$$\mathrm{cost}(\,C\,) \le \mathrm{cost}(\,C_t\,) + \mathrm{cost}(\,C_b\,) + |X_b| - \#\mathrm{rts}(\mathcal{F}_b) + |C_t|$$

$$\le f(m_t, n_t, r\text{-}s\text{-}1) + f(m_b, n_b, s) + n - n_t - (s+1)\cdot n_t +$$

Each node in $\mathcal{F}_t$ has at least $s+1$ children in $\mathcal{F}_b$, and they must all be different roots of $\mathcal{F}_b$.



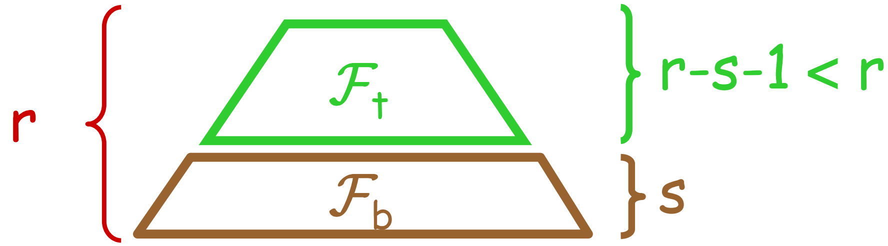
$|X_{>s}| = n_t$ $\qquad$ $|C_t| = m_t$

$|X_{\leq s}| = n_b = n - n_t$ $\qquad$ $|C_b| = m_b$

$$\text{cost}(\,C\,) \leq \text{cost}(\,C_t\,) + \text{cost}(\,C_b\,) + |X_b| - \#\text{rts}(\mathcal{F}_b) + |C_t|$$

$$\leq f(m_t, n_t, r-s-1) + f(m_b, n_b, s) + n - n_t - (s+1)\cdot n_t + m_t$$

Each node in $\mathcal{F}_t$ has at least s+1 children in $\mathcal{F}_b$, and they must all be different roots of $\mathcal{F}_b$.



$|X_{>s}| = n_t$ $\quad |C_t| = m_t$

$|X_{\le s}| = n_b = n - n_t$ $\quad |C_b| = m_b$

$$\mathrm{cost}(\,C\,) \le \mathrm{cost}(\,C_t\,) + \mathrm{cost}(\,C_b\,) + |X_b| - \#\mathrm{rts}(\mathcal{F}_b) + |C_t|$$

$$\le f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + n-n_t - (s+1)\cdot n_t + m_t$$

Each node in $\mathcal{F}_t$ has at least s+1 children in $\mathcal{F}_b$, and they must all be different roots of $\mathcal{F}_b$.

$$f(m,n,r) \le f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t$$

$$f(m,n,r) \leq f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t$$

$$n_t + n_b = n$$

$$m_t + m_b \leq m$$

$$0 \leq s < r$$

$$f(m,n,r) \leq f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t$$

$$n_t + n_b = n \qquad 0 \leq s < r$$

$$m_t + m_b \leq m$$

Assume: $f(\mu,\nu,\rho) \leq k\cdot\mu + \nu\cdot g(\rho)$

$$f(m,n,r) \le f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t$$

$$n_t + n_b = n$$
$$m_t + m_b \le m$$
$$0 \le s < r$$

Assume: $f(\mu,\nu,\rho) \le k\cdot\mu + \nu\cdot g(\rho)$

$$f(m,n,r) \le k\cdot m_t + n_t\cdot g(r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t$$
$$\le k\cdot m_t + n_t\cdot g(r) + f(m_b,n_b,s) + n - s\cdot n_t + m_t$$

$$f(m,n,r) \le f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t$$

$$n_t + n_b = n \qquad 0 \le s < r$$
$$m_t + m_b \le m$$

Assume: $f(\mu,\nu,\rho) \le k\cdot\mu + \nu\cdot g(\rho)$

$$f(m,n,r) \le k\cdot m_t + n_t\cdot g(r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t$$
$$\le k\cdot m_t + n_t\cdot g(r) + f(m_b,n_b,s) + n - s\cdot n_t + m_t$$

choose $s = g(r)$

$$\boxed{f(m,n,r) \le f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t}$$

$$n_t + n_b = n \qquad 0 \le s < r$$
$$m_t + m_b \le m$$

Assume: $f(\mu,\nu,\rho) \le k\cdot\mu + \nu\cdot g(\rho)$

$$f(m,n,r) \le k\cdot m_t + n_t\cdot g(r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t$$
$$\le k\cdot m_t + n_t\cdot g(r) + f(m_b,n_b,s) + n - s\cdot n_t + m_t$$

choose $s = g(r)$

$$f(m,n,r) \le (k+1)\cdot m_t + f(m_b,n_b,s) + n$$
$$\le (k+1)\cdot m_t + f(m_b,n,s) + n$$

$$s = g(r)$$

$$f(m,n,r) \leq (k+1) \cdot m_t + f(m_b,n,s) + n$$

$$s = g(r)$$

$$f(m,n,r) \le (k+1)\cdot m_t + f(m_b,n,s) + n \quad \Bigg| \quad -(k+1)\cdot(m_b+m_t)$$

$$s = g(r)$$

$$f(m,n,r) \leq (k+1)\cdot m_t + f(m_b,n,s) + n \quad \Big| \quad -(k+1)\cdot\overbrace{(m_b+m_t)}^{m}$$

$$s = g(r)$$

$$f(m,n,r) \leq (k+1)\cdot m_t + f(m_b,n,s) + n \quad \Big| \quad -(k+1)\cdot\overbrace{(m_b+m_t)}^{m}$$

$$f(m,n,r) - (k+1)\cdot m \leq f(m_b,n,s) - (k+1)\cdot m_b + n$$

$$s = g(r)$$

$$f(m,n,r) \leq (k+1)\cdot m_t + f(m_b,n,s) + n \quad \Big| \quad -(k+1)\cdot\overbrace{(m_b+m_t)}^{m}$$

$$f(m,n,r) - (k+1)\cdot m \leq f(m_b,n,s) - (k+1)\cdot m_b + n$$

$$\phi(m,n,r) \leq \phi(m_b,n,g(r)) + n$$

$$s = g(r)$$

$$f(m,n,r) \leq (k+1)\cdot m_t + f(m_b,n,s) + n \quad \Big| \quad -(k+1)\cdot\overbrace{(m_b+m_t)}^{m}$$

$$f(m,n,r) - (k+1)\cdot m \leq f(m_b,n,s) - (k+1)\cdot m_b + n$$

$$\phi(m,n,r) \leq \phi(m_b,n,g(r)) + n$$

$$\phi(m,n,r) \leq n \cdot g^*(r)$$

$$s = g(r)$$

$$f(m,n,r) \le (k+1)\cdot m_t + f(m_b,n,s) + n \quad \Big| \quad -(k+1)\cdot\overbrace{(m_b+m_t)}^{m}$$

$$f(m,n,r) - (k+1)\cdot m \le f(m_b,n,s) - (k+1)\cdot m_b + n$$

$$\phi(m,n,r) \le \phi(m_b,n,g(r)) + n$$

$$\phi(m,n,r) \le n \cdot g^*(r)$$

$$f(m,n,r) \le (k+1)\cdot m + n \cdot g^*(r)$$

Shifting Lemma:

If $f(m,n,r) \leq k \cdot m + n \cdot g(r)$

then also $f(m,n,r) \leq (k+1) \cdot m + n \cdot g^*(r)$

Shifting Lemma:

If $f(m,n,r) \leq k \cdot m + n \cdot g(r)$

then also $f(m,n,r) \leq (k+1) \cdot m + n \cdot g^{*}(r)$

Shifting Corollary:

If $f(m,n,r) \leq k \cdot m + n \cdot g(r)$

then also $f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{\overbrace{**\cdots*}^{i}}(r)$

for any $i \geq 0$

If $f(m,n,r) \leq k \cdot m + n \cdot g(r)$

then also $f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{\overbrace{**\cdots*}^{i}}(r)$

for any $i \geq 0$

If $f(m,n,r) \leq k \cdot m + n \cdot g(r)$

then also $f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{\overbrace{** \cdots *}^{i}}(r)$

for any $i \geq 0$

Trivial bound: $f(m,n,r) \leq n \cdot (r-1)$

If $f(m,n,r) \leq k \cdot m + n \cdot g(r)$

then also $f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{\overbrace{**\cdots*}^{i}}(r)$

for any $i \geq 0$

Trivial bound: $f(m,n,r) \leq n \cdot (r-1)$

$= 0 \cdot m + n \cdot (r-1)$

If $f(m,n,r) \le k \cdot m + n \cdot g(r)$

then also $f(m,n,r) \le (k+i) \cdot m + n \cdot g^{\overbrace{**\cdots*}^{i}}(r)$

for any $i \ge 0$

Trivial bound: $f(m,n,r) \le n \cdot (r-1)$

$= 0 \cdot m + n \cdot (r-1)$

$g(r) = r-1$

$g^{*}(r) = r-1$

$$f(m,n,r) \leq f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t$$

$$n_t + n_b = n$$

$$m_t + m_b \leq m$$

$$0 \leq s < r$$

$$f(m,n,r) \leq f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t$$

$$n_t + n_b = n$$

$$m_t + m_b \leq m$$

$$0 \leq s < r$$

Trivial bound: $f(\mu,\nu,\rho) \leq \nu\cdot(\rho-1)$

$$f(m,n,r) \le f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t$$

$$n_t + n_b = n \qquad 0 \le s < r$$
$$m_t + m_b \le m$$

Trivial bound: $f(\mu,\nu,\rho) \le \nu\cdot(\rho-1)$

$$f(m,n,r) \le n_t\cdot(r-s-2) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t$$

$$\boxed{f(m,n,r) \leq f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t}$$

$$n_t + n_b = n \qquad 0 \leq s < r$$
$$m_t + m_b \leq m$$

Trivial bound: $f(\mu,\nu,\rho) \leq \nu\cdot(\rho-1)$

$$f(m,n,r) \leq n_t\cdot(r-s-2) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t$$
$$\leq n_t\cdot(r-2s-4) + f(m_b,n_b,s) + n + m_t$$

$$f(m,n,r) \le f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t$$

$$n_t + n_b = n \qquad 0 \le s < r$$
$$m_t + m_b \le m$$

Trivial bound: $f(\mu,\nu,\rho) \le \nu\cdot(\rho-1)$

$$f(m,n,r) \le n_t\cdot(r-s-2) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t$$
$$\le n_t\cdot(r-2s-4) + f(m_b,n_b,s) + n + m_t$$

set $s = \lfloor r/2 \rfloor$

$$\boxed{f(m,n,r) \le f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t}$$

$$n_t + n_b = n \qquad 0 \le s < r$$
$$m_t + m_b \le m$$

Trivial bound: $f(\mu,\nu,\rho) \le \nu\cdot(\rho-1)$

$$f(m,n,r) \le n_t\cdot(r-s-2) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t$$
$$\le n_t\cdot(r-2s-4) + f(m_b,n_b,s) + n + m_t$$

set $s = \lfloor r/2 \rfloor$

$$f(m,n,r) \le f(m_b,n_b,r/2) + n + m_t$$

$$f(m,n,r) \leq f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t$$

$$n_t + n_b = n \qquad 0 \leq s < r$$
$$m_t + m_b \leq m$$

Trivial bound: $f(\mu,\nu,\rho) \leq \nu\cdot(\rho-1)$

$$f(m,n,r) \leq n_t\cdot(r-s-2) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t$$
$$\leq n_t\cdot(r-2s-4) + f(m_b,n_b,s) + n + m_t$$

set $s = \lfloor r/2 \rfloor$

$$f(m,n,r) \leq f(m_b,n_b,r/2) + n + m_t$$

$$f(m,n,r) - m \leq f(m_b,n_b,r/2) - m_b + n$$

$$f(m,n,r) \le f(m_t,n_t,r-s-1) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t$$

$$n_t + n_b = n \qquad 0 \le s < r$$

$$m_t + m_b \le m$$

Trivial bound: $f(\mu,\nu,\rho) \le \nu\cdot(\rho-1)$

$$f(m,n,r) \le n_t\cdot(r-s-2) + f(m_b,n_b,s) + n - (s+2)\cdot n_t + m_t$$

$$\le n_t\cdot(r-2s-4) + f(m_b,n_b,s) + n + m_t$$

set $s = \lfloor r/2 \rfloor$

$$f(m,n,r) \le f(m_b,n_b,r/2) + n + m_t$$

$$f(m,n,r) - m \le f(m_b,n_b,r/2) - m_b + n$$

$$f(m,n,r) \le m + n\cdot\log r$$

If $f(m,n,r) \leq k \cdot m + n \cdot g(r)$

then also $f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{\overbrace{**\cdots*}^{i}}(r)$

for any $i \geq 0$

If $f(m,n,r) \leq k \cdot m + n \cdot g(r)$

then also $f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{\overbrace{**\cdots*}^{i}}(r)$

for any $i \geq 0$

We know bound: $f(m,n,r) \leq m + n \cdot \log r$

If $f(m,n,r) \leq k \cdot m + n \cdot g(r)$

then also $f(m,n,r) \leq (k+i) \cdot m + n \cdot g^{\overbrace{**\cdots*}^{i}}(r)$

for any $i \geq 0$

We know bound: $f(m,n,r) \leq m + n \cdot \log r$

Therefore for any $i \geq 0$:

$$f(m,n,r) \leq (i+1) \cdot m + n \cdot \log^{\overbrace{**\cdots*}^{i}}(r)$$

For any $i \geq 0$: $f(m,n,r) \leq (i+1) \cdot m + n \cdot \log^{\overbrace{**\cdots*}^{i}}(r)$

For any $i \geq 0$: $f(m,n,r) \leq (i+1)\cdot m + n\cdot \log^{\overbrace{**\cdots*}^{i}}(r)$

Choice of i :

For any $i \geq 0$: $f(m,n,r) \leq (i+1)\cdot m + n\cdot \log^{\overbrace{**\cdots*}^{i}}(r)$

Choice of i :

Define $\alpha(r) = \min\{\, i \mid \log^{\overbrace{**\cdots*}^{i}}(r) \leq i \,\}$

For any $i \geq 0$: $f(m,n,r) \leq (i+1)\cdot m + n\cdot \log^{\overbrace{**\cdots*}^{i}}(r)$

Choice of i :

Define $\alpha(r) = \min\{\, i \mid \log^{\overbrace{**\cdots*}^{i}}(r) \leq i \,\}$

$$f(m,n,r) \leq (m+n)(1+\alpha(r))$$

For any $i \geq 0$: $f(m,n,r) \leq (i+1)\cdot m + n\cdot \log^{\overbrace{**\cdots*}^{i}}(r)$

Choice of i :

Define $\alpha(r) = \min\{\, i \mid \log^{\overbrace{**\cdots*}^{i}}(r) \leq i \,\}$

$$f(m,n,r) \leq (m+n)(1+\alpha(r))$$

$$\leq (m+n)(1+\alpha(\log n))$$

For any $i \geq 0$: $f(m,n,r) \leq (i+1) \cdot m + n \cdot \log^{\overbrace{** \cdots *}^{i}}(r)$

Choice of i :

Define $\alpha(m,n,r) = \min\{\, i \mid \log^{\overbrace{** \cdots *}^{i}}(r) \leq m/n \,\}$

For any $i \geq 0$: $f(m,n,r) \leq (i+1) \cdot m + n \cdot \log^{\overbrace{**\cdots*}^{i}}(r)$

Choice of i :

Define $\alpha(m,n,r) = \min\{\, i \mid \log^{\overbrace{**\cdots*}^{i}}(r) \leq m/n \,\}$

$$f(m,n,r) \leq m(2+\alpha(m,n,r))$$

For any $i \geq 0$: $f(m,n,r) \leq (i+1) \cdot m + n \cdot \log^{\overbrace{**\cdots*}^{i}}(r)$

Choice of i :

Define $\alpha(m,n,r) = \min\{\, i \mid \log^{\overbrace{**\cdots*}^{i}}(r) \leq m/n \,\}$

$$f(m,n,r) \leq m(2+\alpha(m,n,r))$$

Define $\alpha(m,n) = \min\{\, i \mid \log^{\overbrace{**\cdots*}^{i}}(\log n) \leq m/n \,\}$

For any $i \geq 0$: $f(m,n,r) \leq (i+1) \cdot m + n \cdot \log^{\overbrace{** \cdots *}^{i}}(r)$

Choice of i :

Define $\alpha(m,n,r) = \min\{ i \mid \log^{\overbrace{** \cdots *}^{i}}(r) \leq m/n \}$

$$f(m,n,r) \leq m(2+\alpha(m,n,r))$$

Define $\alpha(m,n) = \min\{ i \mid \log^{\overbrace{** \cdots *}^{i}}(\log n) \leq m/n \}$

$$f(m,n,r) \leq m(2+\alpha(m,n))$$